Filosofía de las Matemáticas

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Internalism and Externalism in the Foundations of Mathematics

Apuntes para la sesión del 13 de febrero de 2020

Abstract. Most of the current debate surrounding Internalist an Externalist Theories of Justification is focused on arguments and examples from common or empirical knowledge, while most epistemological discussions on a-priori knowledge focus on conceptual and logical knowledge. Yet, it is easy to see that formalism offers an internalist theory of mathematical knowledge that is both anti-reliabilist, deontological and radically accessibilist. It is anti-reliabilist in so far as it does not presume that our methods of mathematical proof should be reliable or responsive to any external mathematical reality. It is deontological, in so far as it holds that a proof is justification-conferring ultimately because it follows certain epistemic standards and norms. It is extremely accessibilist in so far as further demands that these standards and norms, and the fact that the proof abides by them, be made extremely explicit and clear. It is not surprising, therefore, that many of the criticisms raised by externalists against internalism, in general, and deontological epistemology in particular echo so many criticisms raised against formalism in the philosophy of mathematics. The purpose of this paper is to draw similarities and disanalogies between internalist issues in contemporary epistemology and those of formalist epistemology in mathematics. I will conclude that most of the strongest externalist arguments against internalism become much weaker when charged against formalism.

Without a doubt, one of the main reasons Platonsim remains such a strong contender in the Foundations of Mathematics debate is because of the *prima facie* plausibility of the claim that *objectivity* needs *objects*.[1] It seems like nothing else but the existence of *external* referents for the terms of our mathematical theories and calculations can guarantee the objectivity of our mathematical knowledge. The reason why Frege – and most Platonists ever since – could not adhere to the idea that mathematical objects were mental, conventional or in any other way dependent on our faculties, will or other historical contingencies was that objects whose properties of existence depended on

1. Among the other reasons why Platonism remains an attractive option in the philosophy of mathematics is the possibility of giving a unified semantics for the mathematical and non-mathematical vocabulary in natural language.

such contingencies could not warrant the objectivity required for scientific knowledge. This idea gained currency in the second half of the 19th Century and remains current for the most part today.

However, it was not always like that. Objectivity, after all, has a history, and according to its historians (Daston 2001), the view that scientific knowledge need be objective is a fairly recent one. Up until mid-19th Century, science was not so much concerned with *objectivity,* as it was concerned with *truth*. Before the rise of the modern university and the professional scientist, science had the discovery of *truths* as its ultimate goal. In contrast, modern science now aims at the production and acquisition of *objective knowledge*. The difference might seem subtle, but it is definitely significant. *Truth*, Danston (2001) reminds us is the opposite of *falsehood*: lies, mistakes, fiction, fantasy, confusion, tricks, vagueness, ambiguity, ornament, exaggeration, etc. To search for truth, therefore, is to avoid falsehoods of all these kinds in order to unveil the truth lying behind them. According to Daston, for a long time, the goal of science was to unmask and exhibit the *hidden* or *underlying* truths of nature. Unlike the passive contemplation of nature, science was conceived as a more active search for truths, common to both the explorer and the experimenter. Furthermore, it required a special kind of expertise that came only with experience. Science was a matter of masters and apprentices, much like a craft.

In contrast, the opposite of our current epistemic ideal of *objectivity* is not falsehood but *subjectivity*: individual idiosyncrasy and personal perspective. Objective knowledge requires a strong independence from whatever is proper of the subject, or its historical circumstances. To achieve this objectivity, the scientist, whose expertise and judgment were so valued in earlier times, must be *neutralized*, become *silent, detached*, and take a *view-from-nowhere perspective* on things. Thus, the authority of science was depersonalized and transferred to canonical theories and methods.

This adoption of objectivity as scientific ideal in late XIXth Century meant also a commitment to making science, at least in principle, broadly accessible. The professionalization of science, the establishment of a broad scientific curriculum for basic education, and the development and refinement of mechanical instruments, from the microscope to the digital computer, all pointed in the same direction: towards the democratization of scientific knowledge and science making. Science became no longer something to be acquired with experience, like a craft, but something that could be be learnt from books or classes. The search for explicit rules for the justification of knowledge claims came as an unsurprising corollary to the new paradigm of a science without experts.

Daston's diagnosis, just like those of Gallison, Carey and others historians of objectivity, stems from the study of natural science and its development. However, it takes little to notice that mathematics was not indifferent to this transition. In the late 19th Century, mathematics was also transformed from a science of truths to a science of objective knowledge. As modern mathematics matured and diversified, it became no longer enough for mathematical claims to be *true*. They also had to be objectively grounded. In mathematics, as in the rest of the sciences, it became no longer sufficient to *discover* truths, it was also necessary to demonstrate their objectivity. The so-called "crisis" of the fundaments of mathematics that motivates this volume was nothing but the mathematical embodiment of the new paradigm of objective scientific knowledge. The foundations of mathematics became a pressing problem at the end of the 19th Century not so much because there were doubts about the *truth* of mathematical theories, but because there were doubts about its *objectivity*. In this context, the search for a purely formal mathematical method, with rigorous and explicit rules for demonstration, is nothing but the adoption of the mechanical ideal that Daston identified. The purpose of eliminating intuition and genius (as well as ingenuity) from the heart of mathematics, he formalist turn tried to eradicating al traces of subjectivity from mathematics. The goal was to develop a mathematics without mathematicians, so to speak.

In this context, it is clear why Platonism became such a tempting solution to the problem of the foundations of mathematics: if mathematical truths are determined by objects and relations that are *abstract* – in the strong Fregean sense – that is, imperturbable by any human activity and indifferent to the effects (causal or otherwise) of the human mind, then there is no risk of subjective contamination. There seems to be not better way to keep mathematics off human hands (and minds) than to place it in a distant *third* realm, neither mental, nor physical. However, as is well know, Platonism's radical move may help account for mathematics' objectivity, but also opens a Pandora's box of metaphysical and epistemological problems.

Nevertheless, Dastons' historical analysis already suggests a different strategy. If the problem of the foundations of mathematics is to democratize and mechanize mathematics, in order to generate a rational, broad and stable consensus, then all that is necessary to set mathematics on strong foundations is to make the rules of mathematical knowledge as rigorous, transparent and explicit as possible. Hence, the formalist turn in the foundations of mathematics must also be seen as an attempt to ground the objectivity of mathematical knowledge.[2] Thus, the problem of finding foundations for mathematics, i.e. the problem of demonstrating the *objectivity* of mathematics, gave birth to two closely connected foundational programs: Platonism and Formalism. Platonism kept subjectivity at bay by putting the subject matter of mathematics out of subjective reach. Formalism aimed at the same goal by making the process of doing mathematics (of proving mathematical theorems, specially) the most mechanical and de-personalized possible. In both cases, the goal was to remove all subjectivity from mathematical knowledge.

I want it to be very clear that formalism and Platonism do not offer two alternative or even different *conceptions* of objectivity. On the contrary, they both aim at the same objectivity, following two different paths towards the same goal. The Platonist path starts from the independent existence of mathematical objects, their properties and relations, and then tries to build the objectivity of mathematical knowledge on top of it. Formalism aims at delivering exactly the same sort of objectivity that Platonism does, but from a different starting point: the objectivity of the epistemic

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2. It is not surprising, therefore, that around the problem of the foundations the terms "formalization" and "mechanization" were once used as synonyms. It is also not surprising that works on the foundations of mathematics had also laid the basis for the development of the digital computer.

rules that govern mathematical practices. The purpose of the first section of this text is to show how both approaches converge into a unified conception of objectivity.

My second main claim in this paper is that the main differences in the epistemological approaches to mathematical knowledge and objectivity of Platonists and Formalists also correspond to the main differences between the now-well-known internalist and externalist approaches to justification and knowledge in epistemology. In the second half of the paper, I will try to show this by calling attention to how similar are the criticisms raised by Platonism against Formalism, and the criticisms raised by externalists against internalists. Thus, the final purpose of this paper is to develop the idea of formalism as an internalist alternative to Platonism's externalism in the foundations of mathematics.

I. On objectivity

Recent work in analytic epistemology, philosophy of language and of science has brought back the objective/subjective distinction to the fore (MacFarlane *forthcoming), (*Gallison & Daston 2007), (Wright 2003), (Kölbel 2003, 2000), (Daston 2001), (Searle 1995). As Searle (1995, 8) has clearly stated, the distinction works at different levels and as (Daston 2001) has also stressed, these senses have historically evolved. In this section, I will offer a rough and ready taxonomy of approaches to the objective/subjective distinction, trying to make justice to the contemporary literature on the topic, but also making the necessary adjustment to give a unitary picture. The purpose of this section is to show how the same notion of objectivty can be appoached from so apparently different perspective as those of the Platonist and formalist projects in the philosophy of mathematics. At the end of the section, I hope it becomes clear how Platonism takes an externalist path towards objectivity, while formalism takes an internalist one. For externalism, there is a primacy of the objectivity of mathematical truth (and existence) over the objectivity of mathematical knowledge; for internalism, in contrast, the objectivity of mathematical knowlege is primary and the objectivity of mathetical turth (and existence) is derived from it.

My starting point is Daston's claim that our current understanding of the objective/subjective distinction is based on the identification of particular subjective *factors*, like perspectives, linguistic conventions, psychological architecture, etc. These facts range from (i) the most personal and temporary, like our preferences, attitudes, feelings and perspectives, to (ii) those we share with other members of identifiable social-groups, like the linguistic conventions of a common language and other historical factors, and even (iii) those we share with others because of some common biological properties, like those we may share with people of our same sex or health conditions (Lloyd 1995). Whatever depends on any of these factors is broadly termed "subjective", and only that which is not subjective is called "objective". In order to differentiate between the aforementioned three different sources of subjectivity, it may be useful to talk about (i) "private", (ii) "social" and (iii) "psychological" sources of subjectivity (Swoyer 2008).[3]

Besides these different sources of subjectivity, "Objective" and "subjective" are adjectives that are usually applied to entities of broadly different kinds: knowledge, judgments, (true) propositions, objects and concepts. When applied to objects, subjectivity and objectivity are different modes of existence (Searle 1995, 8). An entity (object or concept) is subjective if its existence depends on one or another subjective factor. Toothaches, baseball teams and colors are all subjective entities, yet their subjective nature is radically different. Pains are private, baseball teams are social (or "institutional"

3. Once this distinction is in place, it is easy to notice that some philosophers draw broader or narrower limits around the subjective. In a very narrow sense, anything besides private subjectivity is considered objective (this, for example, is Searle's position regarding what he calls "epistemic subjectivity" in 1995). Others find social factors as subjective as private ones, but not psychological factors, especially those that are species-specific. For Stephen Stitch (1990), for example, at least some psychological phenomena may be as objective as material ones, even if they are strongly dependent on our biological makeup. In contrast, others, most notably Frege (1884) and many other early analytic philosophers clearly took a strong view of objectivity, where psychological factors were deemed too subjective. (Jacquette 2003)

to use Searle's term), and colors are psychological. They would not exist, were it not for our specific psychological makeup, our personal subjective perspective and our sports institutions.

When talking about propositions, one is called "subjective" if whatever makes it true (or whatever determines whether it is true or false) includes or depends on subjective factors. Analogously, propositions that have determinate truth values independently of any subjective (private, social or psychological) factor are objective. Thus, propositions like "Wheat Oats are delicious with milk", "Austin is the capital of Texas" and "The Sky is Blue" are all subjective truths. The facts that make them true are not objective.[4] However, some are made true by private facts, others by social facts and finally some may be made true by psychological facts (Nagel 1974).

Subjective truths are sometimes also called "relative", because their truth-value is not absolute, but sensitive to subjective factors like perspective, context, etc. Instead of having a determinate truth-value, their truth-value may vary among individuals, moments in time, social groups or even psychological features. Recent philosophy of language has exploited this feature of subjectivity to devise a test for relativity: so-called "context-shifting arguments" (Cappelen and Lepore 2003). The main idea behind these tests is that if the truth value of the proposition expressed by a sentence is relative to subjective features of the context of utterance, like personal features of the speaker (or hearer), its historical and social context or its biological makeup, etc., then it may change truth values if uttered in different contexts. If the truth-value of a sentence shifts in response to

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4. Sometimes people tend to use the term "fact" to refer only to objective facts (for example, Kripke 1982) and not to any subjective factors that make these other kinds of truths true. So, when people talk about facts, they often mean objective facts, unless otherwise stated.

changes in the subjective features of their context of utterance or evaluation, we have good reasons to believe the proposition is subjective.[5]

Besides context-sensitivity to subjective factors, another important phenomenon associated to subjectivity is the existence of so-called "faultless disagreement" (Kölbel 2003) regarding the truth of subjective propositions. Subjectivity makes it is possible for two parties to disagree on the truth value of a given proposition, not because of any substantial fault on the part of the participants (or, to be more precise, no fault in their *inquiry* on the truth of such proposition), but because of the matter under disagreement itself. If the parties in disagreement do not share the subjective features that determine the truth-value of the proposition, then each one of them they may faultlessly take it to have one truth value or the other.[6]

Finally, besides entities and propositions, there is also meaningful talk of subjective or objective *judgments* or *beliefs*. For someone's belief to be subjective, at least one of the grounds upon which the belief is based must be subjective, otherwise the belief is objective. For example, if I base my judgement of the taste of a cigar on subjective aspects of my personal experience smoking it, then my judgement may be rightfully called subjective. In this case, my judgment is clearly subjective, as

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5. As a corollary, just as subjectivity manifests as context-sensitivity, objectivity manifests as contextual insensitivity or invariability. In other words, just as every sentence that expresses a subjective proposition is context-sensitive, every objective proposition is expressible in a context-invariant proposition, i.e. one whose truth value remains stable across contexts (Lycan 1996). But of course, as stated above, not every context-sensitive sentence expresses a subjective proposition and not every context-invariant sentence expresses an objective proposition. "Arthur Barthres is in indescribable pain at 2:19 pm on the 13th of May, 2009" is an invariant sentence, yet expresses a subjective proposition.

6. Even though the term comes from the work of Kölbel, this way of cashing out epistemic objectivity originates in the pragmatism of Charles Peirce (1877), and was recently updated by Crispin Wright (1992). Like Peirce before them, Wright and Kölbel conceive of objectivity as the end result of an idealized rational inquiry, i.e. as agreement between ideal rational inquirers. Theories of objectivity of this kind are called consensus, intersubjective or agreement theories, in contrast to so-called mirroring or correspondence theories of objectivity that hold that the objectivity or subjectivity of propositions depends primarily on the objectivity or subjectivity of what those propositions are about. (Rorty 1979, Gauker 1995)

subjective is the truth of the proposition being judged. Notice, however, that I may still subjectively judge an objective proposition. For example, I may ground my judgment of whether my parent's place is farther from my home than my office at the university (which is clearly an objective matter of fact) on my subjective appreciation of how long the drive to one or the other seems to me.

Just as we can talk about subjective and objective judgment, we can talk about subjective and objective warrant or justification (if the grounds for belief or judgment are warrant or justification conferring). However, it is quite a controversial issue whether there is such a thing as subjective *knowledge* or not. For a subject *S* to subjectively know a proposition *p, S*'s subjective grounds for believing in *p* must be strong enough to qualify as knowledge. For example, I may know subjectively what it is like to be me or to feel the things I do (Nagel 1974), or I may know subjectively how red things look (Jackson 1982). However, for many philosophers subjective grounds can never be strong enough to qualify as knowledge. From a physicalist perspective, for example, if an agent knows a proposition, all his grounds for it must be objective (Dennett 1991).

Once we have drawn the difference between subjective truth and subjective judgment and knowledge, we can determine if they are related and how. In particular, it is important to determine whether objective truths can only be known objectively (if they can be known at all) or not. Above, I have given an example of a subjective judgement regarding an objective truth: someone who judges distance based on her personal perspective on how long it seems to take to get from one place to another. Whether it is possible to find adequate subjective grounds for *knowing* an objective truth or not is still an open question. Yet, for the remaining of the paper I will assume the default position that it is not. I will assume that the necessary grounds for knowing an objective truth cannot depend on subjective factors. In other words, I will assume that all knowledge of objective truths is objective.

If we assume that epistemic objectivity and objective truth are as closely related as I assume they are, then it is possible to derive the former form the later. In particular, it would be enough to show that the subject matter of our mathematical theories is objective to show that our mathematical knowledge is objective as well. If mathematical truth is objective, mathematical knowledge cannot be but objective. This means that we can try showing that mathematical knowledge is objective *directly* or deriving it from the objectivity of it subject matter. However, if we follow this second path, in order to show that our mathematical judgments are objective, it would not be enough to show that mathematical truths are objective as well. It would also be necessary to show that what we call "mathematical knowledge" is actual knowledge of its objective subject matter, i.e. that our mathematical judgments reliably track objective mathematical facts. Thus, we are left with two strategies to show that mathematics is an objective discipline, depending whether we take the objectivity of mathematical knowledge or the objectivity of mathematical truth as primitive. The first strategy entails trying to prove directly that mathematical knowledge is objective,[7] while the second requires dividing the job into two tasks: first, showing that mathematical truths are objective and then showing that our mathematical methods are truth-conducive. In other words, we must show first that what mathematics is about is objective, and then show that our mathematical practices actually deliver factual knowledge of it. From now on, let me call the first alternative "internalism", and the second one "externalism", for they correspond, in more than a rough way, with what contemporary epistemologist call "internalist" and "externalist" theories of justification and knowledge.

II. Internalism and Externalism

Far from the debates on the foundations of mathematics, mainstream analytic epistemology has bred two different brands of theories of justification: *externalism* and *internalism*. For externalist epistemologists a belief is justified[8] "if and only if it is… formed by means of a process that is truth-conducive in the possible world in which it is produced" (Goldman 1988, 56). In other words, someone's beliefs are "justified only if they [are] *in fact* reliably related to the world, whether or not

7. That is, without assuming the objectivity of what mathematical knowledge is about.

8. Strongly justified, in Goldman's terminology.

he had any reason for thinking this to be so" (Bonjour 1980, 14). Externalists ground knowledge and justification in the responsiveness of our beliefs to whatever they are about, i.e. to their reliable faithfulness to the world, independently of any subjective evaluation of evidence or similar internal judgment. Since internalism cannot define knowledge or justification in terms of truth-conduciveness or any similar truth-related notion, in order to maintain the primacy of epistemic objectivity over objective truth, all factors relevant to determine if a subject is in a state of knowledge or justification or whether a cognitive process confers justification or not must be internal to the subject. Consequently, internalism is based on two central theses: *accessibilism* and *anti-reliabilism*.[9]

> [Accessibilism] is a thesis about the basis of either knowledge or justified belief. This first form of internalism holds that a person either does or can have a form of access to the basis for knowledge or justified belief. The key idea is that the person either is or can be aware of this basis. (Externalists, by contrast, deny that one always can have this sort of access to the basis for one's knowledge and justified belief.) (Pappas 2005)[10]

Anti-reliabilism, on the other hand, is a thesis about the methods of either knowledge or justified belief. It holds that a method may be justification-conferring without necessarily being reliable, that is, without having to be responsive to any external world or reality.

Finally, the main thesis of *deontological* epistemologies (recently espoused by Bonjour as late as 1980, but "common to the way philosophers such as Descartes, Locke, Moore and Chisholm have thought about justification." Steup 2005) "is that the concept of epistemic justification is to be analyzed in terms of fulfilling one's intellectual duties or responsibilities." (Pappas 2005) According to deontologists our epistemic duty or responsibility is "to follow the correct epistemic norms [not

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9. Perhaps, it would be better to say that there are two forms of epistemological internalism, depending on whether they accept one thesis or the other

10. Accesibility, so defined, of course, is a modal notion and, as such, it is susceptible of all the criticisms of the explanatory value of modal notions. Mostly, it is gradual. However formalism is an extreme version of accessibilism, as we will see soon, where evidence (and the epistemological norms) must be fully explicit and accessible, that is, strongly accessible to any individual in any subjective circunstance.

just to act in accordance with them, but to be genuinely guided by them].[11] [And] If this answer is going to help us figure out what obligations the truth-aim imposes on us, we need to be given an account of what the correct epistemic norms are" (Steup 2005).[12]

Combining deontologism with an extreme form of accesibilism, we get the thesis that, in order to be justified in one's belief, the fact that one's holding such belief does not break any current epistemic norm (whatever these may be) must be directly accessible to anyone. In other words, both the reasons for one's belief, the epistemic rules that govern them, and the fact that the reasons given constitute genuine justification according to such norms, must be maximally explicit (or at least, it must relatively easy to make them explicit) and clear.

1. Internalism and Externalism in the Philosophy of Mathematics

Taking an externalist approach to mathematical objectivity unavoidably leads to some form of realism (Shapiro 1997), since this later foundational program places the objectivity of mathematical truths (and the objective existence of mathematical entities) as primary, and the objectivity of mathematical knowledge as derivative. Unsurprisingly, this realist externalism, in turn, leads to some variation of Benacerraf's epistemological challenge: how can our mathematical methods of proof (intuition or convention) be responsive to whatever our mathematical truths are supposed to be about, presumably an abstract reality? (Field 1991, Potter 2007)

11. These epistemic norms may be either backwards or forwards looking ones. Backwards looking epistemic norms aim at regulating belied acquisition, while forward looking ones kick in once one's belief is already in place.

12. Sometimes, internalist deontologism is cashed out in terms of the satisfaction of one's own subjective standards (Kornblith 2001). However, this is true only if it is taken to mean that these norms and standards have to be internalized by the subject, not that they have to be "subjective" in any of the senses detailed in the first section of this paper. On the contrary, internalism is based on the idea that objective norms can be internal as well.

Internalism, on the other hand, is present in the formalist turn in the foundations of mathematics, i.e., the idea that to place mathematical knowledge on a firmer basis, we must develop a rigorous, formal description of the basic concepts and methods of mathematics. This foundational strand shows all the characteristic signs of strong internalism: It is anti-reliabilist, deontological and also radically accessibilist. It is anti-reliabilist in so far as it does not presume that our methods of mathematical proof should be reliable or responsive to any external mathematical reality. It is deontological, in so far as it holds that what makes a proof justification-conferring is that it obeys certain epistemic standards and norms. It is extremely accesibilist in so far as it further demands that these standards and norms, and the fact that the proof abides by them, be made extremely explicit and clear.

It is not surprising, therefore, that many of the criticisms raised by externalists against internalism, in general, and deontological epistemology in particular, echo so many of the criticisms raised against the aforementioned formalist turn. In the remaining of the paper, I will try to look into this criticisms from the perspective of the internalist/externalist distinction regarding objectivity and the internalist/externalist debate in epistemology.[13]

2. Formalism as Internalism

I said that I would show that *many* of the criticisms raised against deontologism, echo similar criticisms raised against Formalism. However, not all of them do. For example, "it has been argued (by, among others, Alston 1989) that any deontological theory of justification presupposes that we can have a sufficiently high degree of control over our beliefs" (Steup 2005), higher than we do for

13. Despite the recent resurgence of epistemological interest in a-priori knowledge, most current work on internalist and externalist theories of knowledge and justification focus their arguments and examples on common or empirical knowledge. Little or no mention is made of a priori or mathematical justification. I hope the rest of this paper helps to fill some of this huge gap. Hopefully, the current debates on epistemic justification may throw new light into the epistemology of mathematics and, vice versa, may new epistemological insights be gained from introducing mathematics into the internalist/externalist debate.

most of our beliefs. But this is not a problem for a deontologist epistemology of mathematics, for – for the most part - we do have such degree of control over our scientific beliefs, in general,[14] and mathematical beliefs, in particular. These are not the kind of beliefs (if any) that may just pop into our minds. Except for rare cases – mostly, involving basic arithmetical and geometrical beliefs –, our mathematical beliefs are acquired on a very controlled environment. More than merely acquired, our mathematical beliefs are consciously *accepted*. We may have *hunches* (sometimes also called *intuitions* or *impulses*)[15] regarding whether a certain hypothesis is true or not, but – once again, except for some basic mathematical claims, mathematicians do not ground their beliefs on them. The acceptance of mathematical beliefs is consciously guided by proof.[16] In this regard, externalism is based on the truism that we do not have voluntary control over how we respond to evidence (Feldman 2001). No matter how strong willed, a mathematician cannot face (and understand) a sound proof without also accepting the corresponding theorem. As Alston puts it:

> I could try asserting the contrary in a confident tone of voice. I could rehearse some skeptical arguments. I could invoke the *Vedantic* doctrine of *maya*. I could grit my teeth and command myself to withhold the proposition. But unless I am a very unusual person, none of these will have the least effect (Alston 1989, 129)

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14. That is why externalism is more attractive as an epistemology for common knowledge instead of scientific knowledge (Bonjour 1980, Alston 1986). Furthermore, this has also driven certain epistemologists like Ernest Sosa (2007) and Angeles Eraña (2009) to sustain a dual theory of epistemic justification: externalist for common knowledge (or for rationality, in Eraña's case) and internalist for science

15. However, this is not the way most Platonists understand mathematical intuition. Cf. Katz 2000 and Plantinga 1996.

16. And testimony. But even those cases when we come to believe complex mathematical propositions through testimony are backed up by the existence and publication of proofs. Now, since the public existence of proofs satisfies the deontological conditions for justification, those beliefs are also justified in the internalist sense.

However, the kind of control that is missing in these cases is not the kind required for deontologism to work. As Anthony Booth (forthcoming) argues, deontologism can be adequately grounded on the indirect control of our beliefs.

> We clearly have voluntary control over many things that influence belief, these things include: whether and for how long one considers a particular issue, looks for relevant evidence or reasons, reflects on a particular argument, seeks the opinions of others, and trains one self to be more critical of such things as gossip and the unquestioned word of putative authorities… What such a deontologism will require and prohibit …are certain activities that will influence belief acquisition.

Even if belief is not an activity under our direct control, and neither is our reaction to evidence, we still have strong control over many other activities that influence belief.[17] Mathematical proof is just one of these activities. This sort of indirect control makes us responsible enough for at least some of our beliefs, including mathematical ones (for we have strong control over our proving practices.) At least for them, deontologism may still account for their justification.

Another common criticism of deontological internalism that is also found in discussions on the foundations of mathematics, however, must be taken more seriously. It cuts more deeply, because, if right, it could actually show that internalist justification does not deliver *objectivity*. As stated at the beginning of this text, for knowledge to be objective, it must be independent of any subjective element, either perspectival, socio-historical or psycho-cognitive. However, it is hard to see how a set of explicit formal rules and axioms could achieve such objective status as to serve as a foundation for mathematical knowledge. From this perspective, the main challenge facing formalists in mathematics is to show that *our* epistemic norms are not culturally or cognitively determined or dependent enough to raise the red flag of relativism. Unless formalists want to become conventionalists and, thus, loose their original objectivist motivation, this is a criticism to be taken seriously.

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17. Even though I agree with Booth, I would like to extend his notion of indirect control to cover not only activities that influence belief acquisition, but also those that affect belief maintenance, so to speak. Epistemic responsibility is both backwards and forwards looking.

Once again, a similar criticism has been raised against internalism by externalists. Besides the aforementioned commitment to doxastic voluntarism, externalists commonly demand internalists a justification for the adoption of their epistemic norms and axioms. However, such criticism can be easily rebutted as a *petitio principi* in so far as whatever notion of justification is demanded in the criticism must be either internalist or externalist. If the justification is internalist, the internalist answer can be circular but non-vicious. If the justification required is externalist – for example, if it is further required that the norms that govern our epistemic practices, *i.e.* the rules and axioms of our theories and methods of proof be reliable or truth-conducing, then the request is clearly question begging. In Vahid's words,

> If the problem is to adjudicate between deontological and truth-conductive conceptions of justification, then by taking truth conductivity to be an essential feature of epistemic justification we have already identified the winning side. (1989, 296)[18]

I want to finish this text by mentioning a more recent criticism against epistemological internalism. According to it, internalism is unable to explain the existence of what is known in the epistemological literature as *undercutting defeaters* (Pollock 1986). "Intuitively, where E is evidence for H, an undercutting defeater is evidence which undermines the evidential connection between E and H." (Kelly 2006) So, a mathematical undercutting defeater would be evidence undermining the evidential connection between a proof and whatever it proves. If it is possible for such defeaters to exist in mathematics, the burden of proof on the formalist side would seem to be enormous.[19]

What kind of evidence would count as a mathematical undercutting defeater? Let S be an epistemic agent whose evidence for holding a mathematical belief H is P. If there is another piece of evidence U such that if S has U, then P is no longer evidence of H for S, then U is an undercutting defeater of the evidential connection between H and P. Now, since we are interested in the existence

18. Darragh Byrne has recently made a similar point for a priori justification in general (Byrne 2007, 249-50).

19. I thank Miguel Ángel Fernández for bringing this point to my attention.

of mathematical undercutting defeaters for *formal* evidence, let P be a proof of H.[20] Thus, for U to be an undercutting defeater for the evidential connection between H and P, it must be a piece of evidence such that, if the agent has it, and still bases her belief in H on P, then she is no longer justified. In other words, if mathematical undercutting defeaters of this kind exist, it must be possible for someone to be in possession of a proof for a theorem, base her belief of the theorem on such proof and yet, not be justified in believing such theorem.

For the sake of the argument, assume that such evidence actually exists. If so, then U is either a proof or not. If it is a proof, then either it is a proof of ¬H (or any other proposition inconsistent with H) or a meta-proof that P is not a proof of H (because the formal system is inconsistent, for example). In both cases, we have possible undercutting defeaters. However, it is not difficult for the formalist to make sense of such evidence. After all, we are still talking about proofs. In either case, the formalist can claim that all the so-called undercutting defeater shows was that our previous system of rules was either incomplete or inconsistent. However, both incompleteness and inconsistency can easily be accounted from an internalist formalist perspective.

The interesting cases of undercutting defeaters, if they exist, must come from *external* sources of evidence, not from new proofs. What is required for externalist mathematical undercutting defeaters to exist is for there to be mathematical evidence not based on any proof. That is, it requires that the evidential power of proof be rebutted by something else that is not another proof. Platonists would need to retort to something like *intuition* or another non-formal source of mathematical evidence. As the history of the philosophy of mathematics has shown us, the Platonist's chances are slim.[21]

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20. For the case where H is a basic mathematical belief and E is mathematical intuition, see Kitcher (1983, 2000) and McEvoy (2007).

21. Nevertheless, the attempts at defending mathematical intuition have not ceased. Cf. Maddy (1980), Parsons (1995), Katz (2000), Feferman (2000), Eagle (2008).

Nevertheless, consider the following scenario, based on Kitcher (1983, 2000) and Casullo (1992): a young student of mathematics comes up with a (correct) proof P of a theorem T. He turns it in to his professor who checks it and then (mistakenly) rejects it as incorrect. In this case, it seems that it is *rational* for the student to reject his belief of P, even if it is based on a correct proof. As a matter of fact, it seems that continuing on in his belief would be unreasonably arrogant. Thus it seems that the evidence the professor provides in fact defeats the evidential connection between P and T.[22]

Frank McEvoy considers a similar scenario in (2007), and offers a few replies that may help the internalist camp. On the one hand, he remarks on the importance of the expertise imbalance at the heart of the example. Notice how different our intuitions would be if the mathematician holding the proof was not merely a student, but a professor in tandem with the mathematician challenging his proof. In that case, it would no longer be so rational for him to reject his belief of the theorem. He may want to go back through his proof again, or do something like verifying whether his proof is actually correct or not. But once again, his decision will depend ultimately on formal considerations. As long as formal considerations outweight the epistemological threat posed by social challenges, the formalist need not loose sleep over such challenges (McEvoy 2007, 234).

What happens in the case of the student is not that his proof, when challenged by the professor, no longer serves as evidence for his belief, but that he is *mistaken* in the belief that his belief is *unjustified*. The case shows only that non-experts may *judge* that unsupported testimony serves to defeat the evidential connection between proof and theorem, but not that testimony actually defeats such connection (McEvoy 2007, 233-4). One may find the student who clings to his belief that *p* in the face of the professor's testimony, arrogant but this is surely irrelevant to whether or not he is justified. In McEvoy's own words,

22. Furthermore, notice that what the Professor has provided is not a proof, but merely a testimony, that is, external evidence.

> ...one's belief is not shown to be unjustified merely on the basis of one's arrogance. Since nothing in the case shows that the subject's *justification* is undermined by the misleading [professor], there seems nothing wrong with claiming that the subject is both arrogant and *justified* in his belief. (McEvoy 2007, 234)[23]

As long as the professor does not give proof of his claim that the student's proof is incorrect, the evidential connection between proof and theorem has not been properly challenged. If our professor offers a proof against the student's mathematical belief, the case would be that of one set of formal considerations defeating the evidential power of other formal considerations. But, as stated above, this kind of formal defeating presents no problem for the formalist (McEvoy 2007, 235). Either case, *undercutting defeaters pose no threat to the formalist.*

In this paper I have tried to show that, despite the relative oblivion from issues in the philosophy of mathematics in which recent mainstream analytical epistemology has developed, epistemological debates on the foundations of mathematics have followed a path that parallels similar debates in mainstream analytical epistemology. I hope to have shown that many of the criticisms raised by externalist epistemologists against internalism, in general, and deontological epistemology in particular are structurally similar to the criticisms raised by Platonists against formalism in the philosophy of mathematics, and that formalists can defend themselves with relative ease against them. Sometimes their defense would follow similar paths as those followed by other internalists. Other times, formalists have resources that internalists about other kinds of knowledge do not have. After all, mathematical proof provides a stronger, *sui-generis* kind of internalist justification, one that is not easily defeated by non-question-begging externalist considerations.[24]

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23. McEvoy writes in terms of warrant, instead of justification, so I have changed the quote to match my text.

24. A preliminary version of this text was presented at the international congress "El Problema de los Fundamentos de la Aritmética en la Tradición Analítica" on September 2007 in Mexico City. I am extremely thankful of the support and commentaries of Carlos Álvarez, Anthony Booth, Ángeles Eraña, Miguel Ángel Fernández, Max Fernández de Castro, Carmen Martínez, Sergio Martínez, Ricardo Mena, Silvio Mota Pinto and Lourdes Valdivia.

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