ESSAY 10

Diagrammatic Reasoning in Euclid's *Elements*

Danielle Macbeth

Although the science of mathematics has not undergone the sorts of revolutionary changes that can be found in the course of the history of the natural sciences, the practice of mathematics has nevertheless changed considerably over the two and a half millennia of its history. The paradigm of ancient mathematical practice is the Euclidean demonstration, a practice characterized by the involvement of both text and diagram. Early modern mathematical practice, begun in the seventeenth century, is instead computational and symbolic; it constitutively involves the formula language of arithmetic and elementary algebra that one is taught as a schoolchild even today (see Macbeth [2004] and Lachterman [1989]). Over the course of the nineteenth century, this practice gave way, finally, to a more conceptual approach, to reasoning from concepts, for instance, from the concept of continuity in analysis or from that of a group in abstract algebra (see Stein [1988]). This most recent mathematical practice has naturally brought in its train—at least officially, if not in the everyday practice of the working mathematician—a demand for rigorous gap-free proofs on the basis of antecedently specified axioms and definitions. It has also suggested to many that Euclidean mathematical practice is hopelessly flawed.

But Euclidean geometry is not flawed. Although it has its limitations—not everything one might want to do in mathematics can be done in the manner of Euclid—this geometry has, over the course of its two and a half thousand year history, proved to be an extremely successful, robust, and sound mathematical practice, albeit one that is quite different from current mathematical practice. My aim is to clarify the nature of this practice in hopes that it might ultimately teach us something about the nature of mathematical practice generally. Perhaps if we better understand the first (and for almost the whole of the long history of the science of mathematics the only) systematic and fruitful mathematical practice, we will be better placed to understand later developments.

Euclid's *Elements* is often described as an axiomatic system in which theorems are proven and problems constructed though a chain of diagram-based reasoning about an instance of the relevant geometrical figure. It will be argued here that this characterization is mistaken along three dimensions. First, the *Elements* is not best thought of as an axiomatic system but is more like a system of natural deduction; its Common Notions, Postulates, and Definitions function not as premises from which to reason but instead as rules or principles according to which to reason. Secondly, demonstrations in Euclid do not involve reasoning about instances of geometrical figures, particular lines, triangles, and so on; the demonstration is instead general throughout. The chain of reasoning, finally, is not merely diagram-based, its moves, at least some of them, licensed or justified by manifest features of the diagram. It is instead diagrammatic; one reasons *in* the diagram in Euclid, or so it will be argued.

1 *Axiomatization or System of Natural Deduction?*

In an axiomatic system, a list of axioms is provided (perhaps along with an explicitly stated rule or rules of inference) on the basis of which to deduce theorems. Axioms are judgments furnishing premises for inferences. In a natural deduction system one is provided not with axioms but instead with a variety of rules of inference governing the sorts of inferential moves from premises to conclusions that are legitimate in the system. In the case of natural deduction, one must furnish the premises oneself; the rules only tell you how to go on. The question, then, whether Euclid's system is an axiomatic system or not is a question about how the definitions, postulates, and common notions that are laid out in advance of Euclid's demonstrations actually function, whether as *premises* or as *rules* of construction and inference. Do they function to provide starting points for reasoning, as has been traditionally assumed?[1] Or do they instead govern one's passage, in the construction, from one diagram to another, and in one's reasoning, from one judgment to another? Inspection of the *Elements* suggests the latter. In Euclid's demonstrations, the definitions, common notions, and postulates are not treated as premises; instead they function, albeit only implicitly, as rules constraining what may be drawn in a diagram and what may be inferred given that something is true.[2] They provide the rules of the game, not its opening positions.

Consider, for example, the first three postulates. They govern what can be drawn in the course of constructing a diagram: (i) If you have two points then a line (and only one) may be produced with the two points as endpoints; (ii) a finite line may be continued; and (iii) if you have a point and a line segment or distance, then a circle may be produced with that center and distance. In each case, one's starting point, points, and

[1] The underlying assumption perhaps is that, as both Plato and Aristotle thought, any science, including mathematics is, or should strive to be, axiomatic. Insofar as Euclid's system is a paradigm of science, then, it must be axiomatic.

[2] As we will see, Euclid in fact almost never invokes his definitions, postulates, and common notions in the course of a demonstration. They are nevertheless readily identifiable as warranting the moves that are made.

lines, must be supplied from elsewhere in order for the postulate to be applied. And nothing can be done, at least at first, that is not allowed by one of these postulates. But once they have been demonstrated, various other rules of construction can be used as well. For instance, once it has been shown, using circles, lines, and points, that an equilateral triangle can be constructed on a given finite straight line (proposition I.1), one may in subsequent constructions immediately draw an equilateral triangle, without any intermediate steps or constructions, provided that one has the appropriate line segment. Propositions such as I.1 that solve construction problems function in Euclid's practice as derived rules of construction. Once they have been demonstrated, they can be used in the construction of diagrams just as the postulates themselves.

Euclid's common notions, and again most obviously the first three, again govern moves one can make in the course of a demonstration, in this case in the course of reasoning. They govern what may be inferred: (i) If two things are both equal to a third then it can be inferred that they are equal to one another; (ii) if equals are added to equals then it follows that the wholes are equal; and (iii) if equals be subtracted from equals, then the remainders are equal.[3] These common notions manifestly have the form of generalized conditionals, which is just the form rules of inference must take when they are stated explicitly.[4] Furthermore, in this case as well, theorems, once demonstrated, can function in subsequent demonstrations as derived rules of inference. Once it has been established that, say, the Pythagorean theorem is true (I.47), one may henceforth infer directly from something's being a right triangle that the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle. Indeed, Euclid's *Elements* is so called because the totality of its theorems and constructions provide in this way the elements, rules, for more advanced mathematical work.

Definitions can also license inferences, though, as we will see, they have other roles to play as well. If, for example, a diagram in a proposition contains a circle then the definition of a circle licenses the passage to the claim that its radii are equal. If it contains a trilateral figure,

[3]These are, of course, not formally valid rules of inference; they are instead what Sellars has taught us to call materially valid rules.

[4]That rules of inference are inherently conditional in form and essentially general is argued by Ryle ([1950]).

that is, a figure bounded by three straight lines, all of whose sides are equal then the definition of equilateral triangle licenses one to conclude that the figure is an equilateral triangle.

But not everything that happens in the course of a demonstration is governed by an explicitly stated rule, whether primitive or derived. There are two sorts of cases. First, Euclid draws (explicitly or implicitly) various obviously valid inferences, such as that two things are not equal given that one is larger than the other, despite the fact that the rule governing the passage is nowhere explicitly stated. Furthermore, in order to follow a demonstration in Euclid, one must read various things off the relevant diagrams, again according to rules that seem nowhere to be stated. For example, given two lines that cross, or cut one another, in a diagram, Euclid assumes that there is a point at their intersection.[5] The point of intersection seems simply to "pop up" in the diagram as drawn, and is henceforth available to one in the course of one's reasoning.[6] This happens not once but twice in the very first proposition of Book I of the *Elements*, to construct on a given finite straight line an equilateral triangle.

The demonstration begins with the setting out: Let AB be the given straight line. A statement of what is to be done follows: to construct an equilateral triangle on AB. Then the construction is given:

(C1) With center A and distance AB let the circle BCD be described. (This is licensed by the third postulate, though Euclid does not mention this.)

(C2) With center B and distance BA let the circle ACE be described. (Again, the warrant for this, the third postulate, is not mentioned.)

(C3) From point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined. (This is implicitly

[5]Thomas L. Heath ([1956]: vol. 1, 242) takes this to be an objection: "Euclid has no right to assume, without premising some postulate, that the two circles *will* meet in a point C." The objection is, Heath says, "a commonplace."

[6]I borrow this use of the expression "pop up" from Kenneth Manders, to whom I am indebted for helping me to appreciate just how important this feature of Euclidean diagrams is. See his [1996] and [2008].

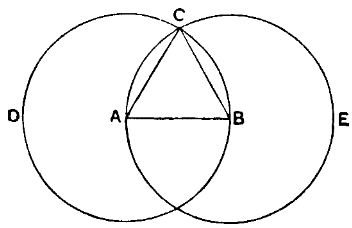


Figure 1

licensed by the first postulate on the assumption that there is such a point C.)

Figure 1 contains the resulting diagram.[7] And the *apodeixis* then follows:[8]

(A1) Given that A is the center of circle CDB, AC is equal to AB. (This is licensed, without mention, by the definition of a circle.)

(A2) Given that B is the center of circle CAE, BC is equal to BA. (This again is implicitly licensed by the definition of a circle.)

(A3) Given that AC equals AB and BC equals BA, we can infer that AC equals BC because what are equal to the same are equal to each other (that is, Common Notion 1).

(A4) Given that AB, BC, and AC are equal to one another, the triangle ABC is equilateral. (This is warranted by the definition of equilateral triangle, *on the assumption that there is such a triangle ABC.*)

This triangle was constructed on the given finite straight line AB as required, and so we are done. In the course of this demonstration, first a point pops up at the intersection of the two drawn circles, and then later a triangle pops up, formed from the radii of the two circles. This

[7]This image is taken from the *Elements*: vol. 1, 241.

[8]The word '*apodeixis*' is generally translated as 'proof.' For reasons that will emerge, I leave it untranslated.

sort of thing is, furthermore, ubiquitous in ancient Greek geometrical practice. One simply reads the relevant geometrical objects off the diagrams, apparently without any explicitly stated warrant for doing so. Although nothing can be put into a diagram that is not licensed by one of the given postulates or a previous construction, there seem to be no stated rules governing what, in the way of pop-up objects, can be taken out of it.[9] We will have occasion to come back to this.

2 Generality in Euclid's Demonstrations

I have suggested that Euclid's *Elements* is not best thought of as an axiomatic system (even one that is incomplete, not fully rigorous) because it seems instead to function more like a system of natural deduction. The Common Notions, Postulates, and Definitions function in the *Elements* not as judgments that will, for the purposes of proof, be taken to be true but instead as rules of passage governing allowable moves in the demonstration. We need now to consider whether the diagram provides an instance about which to reason or instead something more general.

We have seen that Euclid's demonstration that an equilateral triangle can be drawn on a given finite straight line begins with a "setting out" (*ekthesis*): Let AB be the given straight line. And this is generally true of Euclid's demonstrations. Although what is to be demonstrated is something wholly general, the demonstration invariably proceeds by way of such a setting out. To anyone familiar with proofs in standard quantificational logic, it is very easy to take this setting out as an analogue of Universal Instantiation. We read, for instance, in Russell's [1956]:

> Given a statement containing a variable x, say '$x = x$,' we may affirm that this holds in all instances, or we may affirm any one of the instances without deciding as to which instance we are affirming. The distinction is roughly the same as that between the general and particular enunciation in Euclid. The general enunciation tells us some-

[9]Could Euclid's definitions serve this purpose? Can it be inferred, for example, from the fact that a line is a breadless length that at the cut of two lines there is a point (i.e., something that has no parts)? Perhaps, but more would need to be said.

> thing about (say) all triangles, while the particular enunciation takes one triangle and asserts the same thing of this one triangle. But the triangle taken is any triangle, not some one special triangle; and thus although, throughout the proof, only one triangle is dealt with the proof retains its generality. (Russell [1956]: 64)

In quantificational logic, in order to prove that (say) all A is C given that all A is B and that all B is C, one must first turn to an instantiation of the premises in order that the rules of the propositional calculus may be applied. One reasons, in effect, about a particular case, and then at the end of the proof one takes what has been shown to apply generally on the grounds that no inference was drawn in the course of the proof that could not have been drawn were any other instance to have been considered instead. According to Russell, demonstrations in Euclid work the same way.[10] Indeed, one might be puzzled as to what the alternative might be.

Manders offers a suggestion following Leibniz, namely, that "[D]iagrams [are] *textual* components of a traditional geometrical text or argument, rather than semantic counterparts" (Manders [1996]: 391). That is, the diagram does not provide an instance (a semantic counterpart), but instead what Leibniz calls a "character": "For the circle described on paper is not a true circle and need not be; it is enough that we take it for a circle."[11] But what exactly might this mean? Grice's analysis of the distinction between (as he puts it) natural and non-natural meaning provides the tools for at least an outline of a plausible answer (Grice [1957]).[12]

As Grice points out, there is an intuitively clear distinction between, for example, the sense in which a certain sort of spot on one's skin can mean measles and the sense in which three rings on the bell of a bus can

[10] Netz ([1999]: §6) takes essentially the same view: What is to be demonstrated is general but the demonstration itself (that is, the setting out, construction, and *apodeixis*) is particular; its generalizability is "a derivative of [its] repeatability" (ibid.: 246; see also ibid.: 262).

[11] Leibniz [1969]: 84; quoted in Manders [1996]: 291.

[12] See also Dipert [1996].

mean that the bus is full.[13] The task is to provide an analysis of the difference between the two senses of 'means,' natural and non-natural, respectively; and one of Grice's examples to that end is particularly revealing for our purposes. Grice asks us to compare the following two cases:

(1) I show Mr. X a photograph of Mr. Y displaying undue familiarity with Mrs. X.

(2) I draw a picture of Mr. Y behaving in this manner and show it to Mr. X.

And, as he immediately goes on to remark,

> I find that I want to deny that in (1) the photograph (or my showing it to Mr. X) meant$_{\mathrm{NN}}$ [that is, non-naturally] anything at all; while I want to assert that in (2) the picture (or my drawing and showing it) meant$_{\mathrm{NN}}$ something (that Mr. Y had been unduly familiar), or at least that I had meant$_{\mathrm{NN}}$ by it that Mr. Y had been unduly familiar. What is the difference between the two cases? Surely that in case (1) Mr. X's recognition of my intention to make him believe that there is something between Mr. Y and Mrs. X is (more or less) irrelevant to the production of this effect by the photograph . . . But it will make a difference to the effect of my picture on Mr. X whether or not he takes me to be intending to inform him (make him believe something) about Mrs. X, and not to be just doodling or trying to produce a work of art. (Grice [1957]: 282–3)

Although the photograph can serve to convey to someone the fact that Mr. Y is unduly familiar with Mrs. X independent of anyone's intending that it so serve, the drawing cannot. To take the drawing as conveying a message about Mr. Y's behavior, rather than as a mere doodle or as a work of art, essentially involves taking it that someone produced it with the intention of conveying such a message, and that that person

[13]In the opening paragraphs of his essay, Grice lists five different ways the use of 'means' differs in the two cases.

did so with the intention that that intention be recognized (and play a role in the communicative act). Only in that case does the drawing mean (non-naturally) that Mr. Y is unduly familiar with Mrs. X.

A photograph has natural meaning in virtue of a (causally induced) resemblance between the image in the photograph and that of which it is a photograph. A drawing, Grice argues, can in certain circumstances have instead non-natural meaning. That is, it can have the meaning or content that it does in virtue of one's intending that it have that meaning or content (and intending that that intention be recognized and play a certain role in the communicative act). So, we can ask, does a drawn figure in Euclid have Gricean natural meaning or instead Gricean non-natural meaning? If it is a drawing of an instance, a particular geometrical figure, then it has natural meaning. It is in that case a semantic counterpart; it is the thing, say, the particular triangle ABC, that is referred to in the text of a demonstration when one judges of triangle ABC that it is thus and so. But perhaps the drawing, like Grice's drawing, instead has non-natural meaning. Perhaps it is not (say) a circle but instead, as Leibniz says, is taken for one. If so, then the Euclidean diagram can mean or signify some particular sort of geometrical entity only in virtue of someone's intending that it do so and intending that that intention be recognized. One's intention in making the drawing—an intention that can be seen to be expressed in the setting out and throughout the course of the construction—is, in that case, indispensable to the diagram's playing the role it is to play in a Euclidean demonstration.[14]

A Euclidean diagram can be interpreted either as having natural or as having non-natural meaning in Grice's sense. If it has natural meaning then it does so in virtue of being an instance of a geometrical figure. But if it has non-natural meaning then it can be inherently general, a drawing of, say, an angle without being a drawing of any

[14] Jody Azzouni makes a related suggestion in his [2004]. He writes that "[T]he proof-relevant properties of diagrammatic figures . . . are not the actual (physical) properties of diagrammatic figures, but conventionally stipulated ones the *recognition* of which is *mechanically executable* . . . [T]he situation is exactly the same with games: Pieces are conventionally endowed with powers (e.g., to 'take' other pieces under certain circumstances) . . . [A]ll powers a game piece is stipulated to have are mechanically recognizable" (ibid.: 125).

angle in particular. The geometer draws some lines to form a rectilineal angle, in order, say, to show that an angle can be bisected. The geometer does not mean or intend to draw an angle that is right, or obtuse, or acute. He means or intends merely to draw an angle. Hence he says (I.9) that the angle he draws is "the given rectilineal angle," not also that it is right, or obtuse, or acute. That which he draws, regarded as something having natural meaning, will necessarily be right, or acute, or obtuse; but regarded as having non-natural meaning, it will be neither right nor acute nor obtuse. It will simply be an angle. One can no more infer on the basis of the drawing so conceived that the angle in question is (say) obtuse, based on how it looks, than in Grice's case one could infer (say) that Mr. Y has put on a little weight recently, based on how the drawing of him looks. A drawing of Mr. Y could have as its non-natural meaning that he has put on weight recently, just as in the context of a Euclidean demonstration a drawing of an angle can have as its non-natural meaning that it is (say) obtuse, but that would require, in both cases, that quite different intentions be in play.

If the figures drawn in a Euclidean diagram have non-natural rather than natural meaning, then they can, by intention, be essentially general. It seems furthermore clear that they would in that case function as icons in Peirce's sense, rather than as symbols or indices, because they would in some way *resemble* that which they signify.[15] But the resemblance is not, at least not merely and not directly, a resemblance in appearance. As Peirce notes, "[M]any diagrams resemble their objects not at all in looks; it is only in respect to the relations of their parts that the likeness consists" (Peirce [1931]: 282). We can, then, think of a drawn figure in Euclid as an icon that (though it may also resemble its object in appearance) signifies by way of a resemblance, or similarity, in the relations of parts, that is, in virtue of a homomorphism. For example, on such an account a drawn circle serves as an icon of a ge-

[15]Peircean icons can have either natural or non-natural meaning. In particular, individual instances of geometrical figures can be icons of the relevant sorts of things; they have in that case natural meaning. A drawn circle regarded as an instance of a circle is an icon of a circle that has natural meaning because it so functions independent of anyone's intention that it do so. But a drawn circle can also function as an icon with non-natural meaning. In that case it can be essentially general, an icon of a circle not further specified.

ometrical circle not in virtue of any similarity in appearance between the two but because there is a likeness in the relationship of the parts of the drawing, specifically in the relation of the points on the drawn circumference to the drawn center, on the one hand, and the relation of the corresponding parts of the geometrical figure, on the other.

A drawn circle is roughly circular; it looks like a circle just as a dog looks like a dog. But a dog looks like a dog because it *is* a dog, that is, a particular instance of doggy nature (as we can think of it). A drawn circle, I have suggested, can look like a circle for *either* of *two* reasons. It can look like a circle for the same reason that a dog looks like a dog, namely, because it is a circle, a particular instance of circle nature. Or it can look like a circle because it is a icon with non-natural meaning that is intended to resemble a circle first and foremost in the relation of its parts. Because what it is an icon of is circle nature, and because what is essential to a circle's being a circle is that all points on the circumference are equidistant from the center, and it is this relationship of parts that is to be iconically represented, the icon itself comes to look roughly circular. The appearance of circularity is induced in this case by the intended higher order resemblance rather than being something that is there in any case (as circularity is there in any case in a drawing of a particular instance of a circle).

Now if drawings in Euclid function as icons with non-natural meaning, this would explain why Euclid never mentions either straightedge or compass—which one might expect him to if it was an instance that was to be drawn. If one wishes to draw a particular straight line then the best way to do that is with a straightedge, and similarly in the case of a circle: If one wants to draw a particular circle then one is best off using a compass. If, however, one's aim is to draw something with non-natural meaning, in particular, an icon, say, of a line or circle, then all that matters is that one's drawing is able to produce the desired effect, to convey one's intention. In this case there is no reason at all to mention some particular means of producing the drawing precisely because the drawing, to serve the purpose it is to serve need not *look* very much like that for which it is an icon. This is not true of an instance: An instance ought as far as possible to look like what it is. It follows that instances are harder to draw than icons, and this is in fact generally true. It is, for instance, much easier to draw stick human figures and "smiley" faces, which are of course inherently general, than it is to draw someone in particular, the way some particular person actually looks. Even very small children can do the former; most of us even as adults cannot do the latter very well at all.[16] There is no need, then, for mechanical aids in drawing Euclidean diagrams if those diagrams function as has been suggested here, as essentially general icons with non-natural meaning.

More interestingly, the idea that diagrams are iconic and general by intention would also explain why ancient Greek mathematicians almost never make the sorts of mistakes they would be expected to make if their reasoning were based instead on an instance. If one were reasoning about an instance, it would be critical carefully to distinguish between what can and cannot be inferred on the basis of one's consideration of that instance. And this ought, in principle, to be quite difficult. It is, for instance, much harder to learn what a particular person looks like (so as to be able to re-identify that person) from a chance photograph than it is to find this out from, say, a caricature because in the caricature the work of discovering what are the salient and characteristic features of the person's appearance has already been done. Similarly, if a Euclidean drawing were an instance (with natural meaning), it would be hard to distinguish between what Manders has called co-exact features, ones on the basis of which an inference can be made, and exact features, which have no implications for one's course of reasoning (see Manders [1996]: 392–3). But in fact, all the evidence suggests that this is not hard at all. As Mueller remarks regarding the familiar diagram-based "proof" that all triangles are isosceles, "perhaps a 'pupil of Euclid' might stumble on such a proof; but probably he, and certainly interested mathematicians, would have no trouble in figuring out the fallacy on the basis of intuition and figures alone. And in the history of Euclidean geometry no such fallacious arguments are to be found. There are indeed many instances

[16] Notably, some very young autistic children lacking essentially all linguistic ability are able to draw very realistically, that is, things as they actually look, an ability that deteriorates as they learn language. This suggests that we generally draw not what things look like but instead something essentially general, something more like an icon of a concept. See Selfe [1977].

of tacit assumptions to be made, but these assumptions were always true. In Euclidean geometry . . . precautions to avoid falsehood are really unnecessary" (Mueller [1981]: 5). This is unsurprising if the figures in the diagram function not as instances but as icons (with non-natural meaning) that are inherently general. The features that Manders identifies as exact are no more there in the diagram than information about the relative size of human limbs is there in a stick figure.

We saw that drawings of geometrical figures conceived as icons (with non-natural meaning) can resemble particular instances in appearance in virtue of the fact that there is a resemblance in the relation of the parts. A drawn circle, intended as an icon of a circle, can look very much like a particular instance of a circle. It is easy, then, based on such an appearance, to think that the drawing is such an instance, not such an icon at all. There is good reason nonetheless for thinking that in Euclidean demonstrations the diagram functions iconically (i.e., non-naturally) rather than to provide an instance about which to reason. First, as Manders has noted, the latter idea "seems incompatible with the use of diagrams in proof by contradiction" (Manders [1996]: 391). To demonstrate, for instance, that a circle does not cut a circle at more than two points, one first sets out that two circles do cut at more than two points, say, at four. This is not a situation that can obtain; there is no instance to be drawn. And yet the diagram is drawn (as we will see). If the diagram is read instead as a Peircean icon with non-natural meaning there is no difficulty. What the diagram means non-naturally, the content it exhibits, is exactly what one aims to show is impossible, that a circle cuts a circle at more than two points.

Nor is this the only sort of case in which it is impossible to draw an instance. We know, because Euclid tells us in the opening section of the *Elements*, that a point is that which has no parts, and that a line segment is a length that has no breadth (the extremities of which are points). Such entities are clearly not perceptible; there is nothing that a thing with no parts, or a length with no breadth, looks like. It follows directly that there is no way to draw an instance of either a point or a line. On the other hand, such things, and their relations one to another, can be iconically represented. A drawn line length, for example, can represent (be intended iconically to signify) a line with endpoints. A drawn dot can represent (be intended iconically to signify) a point. A drawn circle is, again, a slightly different case because drawn circles do look roughly circular; that is, there is a look that geometrical circles can be said to have. But as in the case of Grice's drawing, the role of a drawn circle in the context of a Euclidean demonstration is not, on the account we are pursuing, that of an instance but instead that of an icon with non-natural meaning, one that is intended to represent the relation of the points on the circumference to the center, the fact that those points are equidistant from the center (whether or not in the figure as drawn the points on the circumference *look* equidistant from the center). All that is represented in a Euclidean straight line (conceived as an icon) is a breadthless length lying evenly with the points on itself, that is, a certain relationship between the line and the points that may be found on it; and similarly, all that is represented in a Euclidean circle (similarly conceived) is the relation of the points on its circumference to the center. In each case, what is important for the cogency of the demonstration is not what the figure looks like but instead what is intended by it whether as set out in Euclid's definitions and postulates, or as stipulated by the particular problem or theorem in question. The similarity in appearance (say, between the icon of a circle and an actual instance of something circular) does help to convey the intended meaning, just as it does in the example of Grice's drawing; nevertheless, what is meant is carried by the intention, not by the similarity in appearance. It is for just this reason that the demonstration can be seen to be essentially general throughout.[17] What one draws in Euclidean diagrams are not pictures of

[17] This is, furthermore, consistent with the final step in the demonstration. As Ian Mueller explains, "[I]n ancient logic the *sumperasma* is the conclusion of an argument. In the Elements, however, the *sumperasma* is not so much a result of an inference as a summing up of what has been established" (Mueller [1974]: 42). As he notes in his [1981], "[T]he word sumperasma [which marks the last stage of a Euclidean demonstration] can . . . mean 'completion' or 'finish' . . . [T]he *sumperasma* merely sums up what has taken place in the proof . . . It merely completes the proof by summarizing what has been established" (ibid.: 13). So conceived, the move from a claim involving letters to the explicitly general claim at the end of a Euclidean demonstration is essentially a move from an implicit generality, such as that a horse is warm-blooded (inferred perhaps from the claim that a horse is mammal together

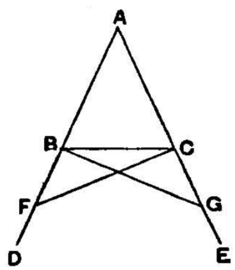


Figure 2

geometrical objects but the relations that are constitutive of the various kinds of geometrical entities involved. As the point might be put, a Euclidean diagram does not *instantiate* content but instead *formulates* it.

3 Diagrammatic Reasoning in Euclid's Elements

We have seen that Euclid's system is best thought of on analogy with a system of natural deduction as it contrasts with an axiomatization. And I have argued that figures in Euclidean diagrams do not provide instances but instead should be read as having non-natural meaning and to function, in particular, as icons. Diagrams so conceived are constitutively general. What remains to be shown is that the reasoning involved in a Euclidean demonstration is not merely diagram-based but instead diagrammatic.

It is clearly true that in a Euclidean demonstration at least some steps in the chain of reasoning are in some way licensed by the diagram. Consider, for example, proposition I.5, that in isosceles triangles the angles at the base are equal to one another, and if the equal straight lines be produced further, the angles under the base will be equal to one another. The diagram is shown in Figure 2.[18] It is then argued that because AF is equal to AG and AB to AC, the two sides FA,

with the fact that mammals are warm-blooded) to the explicit generality that all horses are warm-blooded.

[18] The image is taken from Euclid: vol. 1, 251.

AC are equal to the two sides GA, AB, and they contain a common angle FAG. This inference, from the two equalities, AF to AG and AB to AC, to the double conclusion, first that the two sides FA, AC are equal to the two sides GA, AB, and also that both those same pairs of sides contain the angle FAG, might be thought to be diagram-based in the following sense. Having, first, drawn an isosceles triangle with sides AB and AC equal, and then having produced the straight lines BD and CE by extending (respectively) AB and AC, and taken F on BD at random and cut AG off from AE to equal AF, one has thereby inevitably produced not only FA, AC equal to GA, AB, which is already more or less given, but also the angle FAG, which, as inspection of the diagram shows, is identical both to the angle FAC and to the angle GAB. It is no more possible to construct the required diagram without realizing this identity of angles than it is to embed one circle in a second circle and the second in a third without thereby embedding the first in the third. (This is of course the principle behind Euler diagrams, which can be used to exhibit the validity of various valid syllogistic forms of reasoning, here *barbara*.) So, just as one can infer on the basis of an Euler diagram that some conclusion follows, so one can infer on the basis of the above diagram that the relevant claims are true. The diagram-based inference is, in both cases, valid, that is, truth-preserving, and it is truth-preserving precisely because there exists a homomorphism, a higher level formal identity, that relates relations among elements in the diagram with relations among the entities represented by those elements.

Reasoning using Euler diagrams provides a paradigm of what we might mean by diagram-based inference. In that case, one draws icons of things of a given sort, in particular, circles to represent collections of objects, and furthermore draws them in spatial relations that are homomorphic to relations of inclusion and exclusion over collections. If, for example, all dogs are mammals then one draws the circle representing the collection of dogs inside the circle representing the collection of mammals. If no fish are mammals then one draws the circle representing the collection of fish wholly separate from that representing the collection of mammals, without any overlap at all. Then one can just read off the diagram what the relationship is between the collection of dogs and the collection of fish: The relevant icons are wholly disjoint so one infers that no dogs are fish. The inference, the passage from the premises about dogs, mammals, and fish, to the conclusion about dogs and fish is licensed by the diagram one draws. It is diagram-based. Do diagrams Euclidean demonstrations function in the same way as the basis for an inference? We will see that they do not.

The first indication that diagrams in Euclid function differently from Euler diagrams is the fact that although Euler diagrams can make something that is implicit in one's premises explicit, they cannot in any way extend one's knowledge. Euclidean demonstrations, by contrast, do seem clearly to be fruitful, real extensions of our knowledge. And the reason they are is surely connected to the fact that objects can pop up in a Euclidean diagram. We find out that an equilateral triangle can be constructed on a straight line only because a triangle pops up in the course of the construction. Before that point there is nothing whatever about any triangles in anything we have to work with in the demonstration. Quite simply, there is no sense in which an equilateral triangle is *implicit* in a line, even given Euclid's axioms, postulates, and definitions. Nevertheless, as I.1 shows, such a triangle *is* there *potentially*: given what Euclid provides us, we can construct an equilateral triangle on a given straight line. The triangle just pops up when we draw certain lines. And this is true generally in ancient Greek mathematics: One achieves something new largely in virtue of this pop-up feature of its diagrams. To understand the role of diagrams in Euclidean demonstrations, then, we need to understand these pop-up objects.

In a Euclidean demonstration, what is at first taken to be, say, a radius of a circle is later in the demonstration seen as a side of a triangle. But how *could* an icon of one thing become an icon of another? How, for example, could an icon of a radius of a circle *turn into* an icon of a side of a triangle? Certainly nothing like this happens in the course of reasoning about an Euler diagram. The iconic significance of the drawn circles is fixed and unchanging throughout the course of reasoning. If the lines one draws in a Euclidean diagram had this same sort of iconic significance, the reasoning would clearly stall because in that case no new points or figures *could* pop up. It is also obvious that nothing could possibly be *both* an icon of (say) a radius of a circle and an icon of a side of a triangle at once, because these are incompatible. An icon of a radius essentially involves reference to a circle; radii are and must be radii of circles. An icon of a side of a triangle makes no reference to a circle. So nothing could at once be an icon of both. But one and the same thing *could serve* now (at time t) as an icon of a radius, and now (at time $t' \neq t$) as an icon of a side. The familiar duck–rabbit drawing is just such a drawing; it is a drawing that is an icon of a duck (though of course no duck in particular) when viewed in one way and an icon of a rabbit (no one in particular) when viewed in another.

Some drawings, such as the duck–rabbit, can be seen in more than one way. It would seem, then, that we can say that in the diagram of I.1 one sees certain lines now as icons of radii, as required to determine that they are equal in length, and now as icons of sides of a triangle, as required in order to draw the conclusion that one has drawn an equilateral triangle on a given straight line.[19] The cogency of the reasoning clearly requires both perspectives. But if it does then the Euclidean diagram is functioning in a way that is very different from the way an Euler diagram functions. Much as in the case of the duck–rabbit drawing, and by contrast with an Euler diagram, various collections of lines and points in a Euclidean diagram are icons of, say, circles, or other particular sorts of geometrical figures, *only when viewed a certain way*, only when, as Kant would think of it, the manifold display (or a portion of it) is synthesized under some particular concept, say, that of a circle, or of a triangle.[20] The point is not that the drawn lines underdetermine what is iconically represented, as if they have to be supplemented in some way. It is that the drawing as given, as certain marks on the page, has (intrinsically)

[19] Euclid's definitions would seem to play a crucial role here: Only if one can see in the diagram that the definition is satisfied can one find the relevant object in it.

[20] This idea, that a system of signs may function not by way of combinations of primitive signs that have designation independent of any context of use, but instead by appeal to primitive signs that designate only given a context of use, is appealed to also in Sun-Joo Shin's ([2002]) reading of Peirce's notation for his alpha logic. In my [2005], I argue that Frege's notation functions in an essentially similar way. The primitive signs of the language express senses independent of their occurrence in a sentence but designate only in the context of a sentence and relative to an analysis, relative, that is, to a way of regarding the whole two-dimensional array of signs.

the *potential* to be regarded in radically different ways (each of which is fully determinate albeit general, again, as in the duck–rabbit case). It is just this potential that is actualized in the course of reasoning, as one sees lines now as radii and now as sides of a triangle, and which begins to explain the enormous power (and beauty) of Euclidean demonstrations as against the relative sterility (and lumpishness) of Euler diagrams.

In Euclidean diagrams, geometrical entities, points, and figures, pop up as lines are added to the diagram. Cut a line AB with another line CD and up pops a point E as the point of intersection, as well as four new lines AE, BE, CE, and DE; take the diagonal of a square and up pop two right triangles, and so on. We do not find this surprising in practice; indeed, one need have no self-conscious awareness that it is happening as one follows the course of a Euclidean demonstration. Nevertheless, as we have seen, it clearly *is* happening, and the cogency of the demonstration essentially depends on it. Such pop up objects, I have suggested, depend in turn on our capacity to "re-gestalt" various collections of lines, to see them now one way and now another. And what this shows is that it is not the lines themselves that function as icons (even in light of one's intention that they be so regarded) but only the lines *when seen from a particular perspective*, when viewed in one way rather than another equally possible way.

Consider again our two crossed lines AB and CD that cut at E, and suppose that they are functioning as icons independent of any perspective taken on them. One could then argue that the point E can belong to only one of the four segments AE, BE, CE, or DE, leaving the other three without an endpoint at one end. This would be a perfectly reasonable inference if we were dealing with a simple icon, one whose significance was fixed independent of any perspective taken on it, because it is true that if you divide a (dense) line by a point then that point can be the endpoint of only one of the two line segments, leaving the other to approach it indefinitely closely (because the line is dense), without having it as its endpoint. Suppose now that we offered this argument to an ancient Greek geometer. How would he respond? He would laugh us away, much as he laughs away those who would "[attempt] to cut up the 'one'" by pointing out that the line drawn to iconically represent the unit can surely be divided.[21] The geometer laughs because one is in that case taking the diagram the wrong way, because one fails to understand how it works as a diagram.

When two lines AB and CD cut at E, only one point pops up and it is the endpoint of all four lines AE, BE, CE, and DE, which also just pop up with the cut. And they can do because E is the endpoint of any one of these lines only relative to a way of regarding it, much as certain lines in the duck–rabbit drawing are ears, or a duckbill, only relative to a way of regarding those lines. Another example makes the same point. Suppose that I claimed on the basis of the drawing of a straight line segment ABCD that, contrary to what Euclid holds, two straight lines *can* have a common segment: AC and BD have BC in common. Again the geometer would laugh—and rightly so. Of course one can see the drawn line as iconically representing AC, or BD, or AD, or BC, and indeed one may need so to regard it in the course of a demonstration; but that does not show that the lines themselves have a common segment. This would be required only if the drawn line functioned iconically to represent these possibilities independent of any perspective that was taken on the drawing. Between two points there is only one straight line, not many; two points can seem to have a common segment (as above) only if one, perversely, mistakes the way the diagram functions.

I have been arguing that the lines in a Euclidean diagram, like the lines in a duck–rabbit drawing, function as icons of various sorts of geometrical objects only relative to a perspective that is taken on them. But those diagrams have a further feature as well, one that is not found either in the duck–rabbit drawing or in an Euler diagram, a feature that is essential to the fruitfulness of Euclidean demonstrations. Euclidean diagrams are distinguished in having three levels of articulation as follows. At the lowest level are the primitive parts, namely, points, lines, angles, and areas. At the second level are the geometrical objects we are interested in, those that form the subject matter of geometry, all of which are wholes of those primitive parts. At this level we find points as endpoints of lines, as points of intersection of lines, and as centers of circles; we find angles of various sorts that are limited by lines that are

[21] Plato, *Republic Book VII*: 525e.

also parts of those angles; and we find figures of various sorts. A drawn figure such as (say) a square has as parts: four straight line lengths, four points connecting them, four angles all of which are right, and the area that is bounded by those four lines. Of course, in the figure as actually drawn, the lines will not be truly straight or equal, and they will not meet at a point; the angles will not be right or all equal to one another. But this does not matter because the drawn square is not a picture or instance of a square but instead an icon of a square, one that formulates certain necessary properties of squares. At the third level, finally, is the whole diagram, which is not itself a geometrical figure but within which can be discerned various second-level objects depending on how one configures various collections of drawn lines within the diagram. What we will see is that these three levels of articulation, combined with the possibility of a variety of different analyses or carvings of the various parts of the diagram, account for the ways in which one can radically reconfigure parts of intermediate wholes into new (intermediate) wholes in the course of a Euclidean demonstration, and thereby demonstrate significant and often surprising geometrical truths.

As already noted, a Euclidean demonstration comprises a diagram and some text, most importantly for our current purposes, the *kataskeue* (construction) and the *apodeixis*. The *kataskeue* provides information about the construction of the diagram and is governed by what can be formulated in the diagram as legitimated by the postulates and any previously demonstrated problems. The *apodeixis*, which is governed by what can be read off the diagram as legitimated by the definitions, common notions, and previously demonstrated theorems, should be read, on our account, *not* as the medium of the proof but instead as providing instructions regarding how various portions of the constructed diagram are to be read, construed, or analyzed, that is, how they are to be carved up in the course of the demonstration. It is the *diagram* that is the site of reasoning, on our account, not the accompanying text.

Consider one last time the first demonstration in Euclid, that on a given straight line an equilateral triangle can be constructed. The diagram is displayed, again, in Figure 3. According to the reasoning already rehearsed, we know that AC = AB because A is the center of

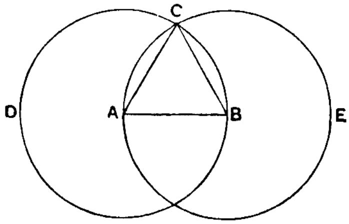


Figure 3

circle CDB, and that BC = BA because B is the center of circle CAE, hence that AC = BC = AB, on the basis of which it is to be inferred that ABC is an equilateral triangle constructed on the given line AB, which was what we were required to show. Our task concerning this little chain of reasoning is to understand how, based on a claim about radii of circles, one might infer, even given the diagram, something about a triangle. Were the reasoning merely diagram-based, that is, were it reasoning *in* (natural) language but, at least in some cases, *justified by* what is depicted in the diagram then the problem seems intractable. No mere diagram of some drawn circles, however iconic, can justify an inference from a claim about the radii of circles to a claim about a triangle. If, on the other hand, we take the diagram not merely as a collection of icons of various geometrical entities but instead as a display with the three levels of articulation outlined above, and so as a collection of lines various parts of which can be configured and reconfigured in a series of steps (as scripted by the *apodeixis*), the solution of our difficulty is obvious. One considers now one part of the diagram in some particular way and now another in some (perhaps incompatible, but perfectly legitimate) way as one makes one's passage to the conclusion. In particular, a line that is at first taken iconically to represent a radius of a circle is later taken iconically to represent a side of a triangle. It is only because the drawing can be regarded in these different ways that one can determine that certain lines are equal, because they can be regarded as radii of a single circle, and then *conclude* that a certain triangle is equilateral, because its sides are equal in length. The demonstration is fruitful, a

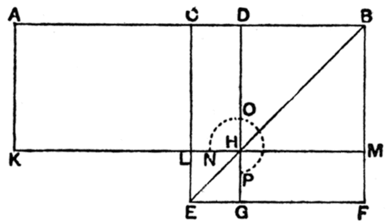


Figure 4

real extension of our knowledge, for just this reason: Because we were able to take a part of one whole and combine it with a part of another whole to form an utterly new and hitherto unavailable whole, we were able to discover something that was simply not there, even implicitly, in the materials with which we began.

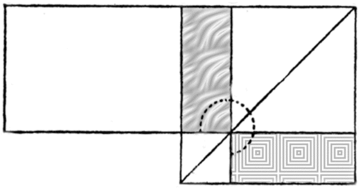
4 More Examples

Consider now the Euclidean demonstration of Proposition II.5: If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half. The diagram is shown in Figure 4.[22] We letter it for ease of reference in our written discourse; were the presentation an oral one would need only to point to relevant aspects of the drawn diagram. To understand this diagram, that is, to know how to "read" it, configure its parts (at least to begin with), we need to know the provenance of its various parts. We are given that line AB is straight, and that it is cut into equal segments by C and into unequal segments by D. We know by the construction that CEFB is a square on CB, which we know from proposition I.46 is constructible, and that DG is a straight line parallel to CE and BF, KM parallel to AB and EF, and AK parallel to CL and BM, shown to be constructible in proposition I.31. That is, already we see here the various ways the drawn lines are taken to figure in various iconic representations of geometrical figures. CB, for example, is first

[22]The image is taken from Heath [1956]: vol. 1, 382.

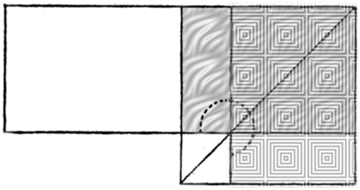
taken to be the half of line AB, but then as a side of a square. BM is a line length equal in length to DB but also a part of the line BF, which is another side of that same square. It is in virtue of these various relations of parts iconically represented in the drawn diagram, and of these reconfigurings of parts, that the diagram can show that the theorem is true. It does so in a series of six steps scripted by the *apodeixis*:

(1)



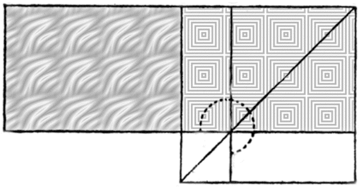
These (shaded) areas are equal, as is shown by Prop. I.43: The complements of a parallelogram about the diameter are equal to one another.

(2)



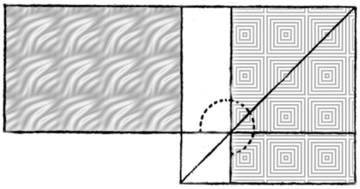
It follows from (1) that these areas are equal, because the same has been added to the same.

(3)



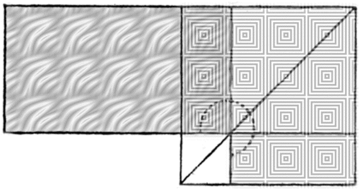
These areas are known to be equal on the basis of what we know about the relationships that obtain among the lines that form the boundaries of the two shaded areas.

(4)



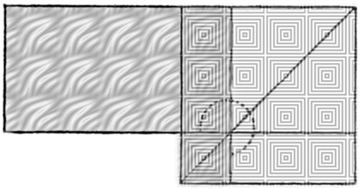
It follows from (2) and (3) that these areas are equal because things equal to the same are equal to each other.

(5)



It follows from (4) that these areas are the same because the same has been added to equals.

(6)



It follows from (5) that these areas are equal because the same has been added to equals.

But that is just what we wanted to show: that the square on the half is equal to the rectangle contained by the unequal segments plus the square on the line between the points of section. By construing various aspects of the diagram in these various ways in the appropriate sequence one comes to see that the theorem is, indeed, must be, true. But if that is right, then (again) it is the diagram that is the site of reasoning in Euclid, not the text. The text, on this account, is merely a script to guide one's words, and thereby one's thoughts, as one walks oneself through the demonstration.[23] The demonstration is not in the words of the *apodeixis*; it is in what those words help us to see in the diagram.[24]

[23]Think of doing a calculation in Arabic numeration, say, working out the product of thirty-seven and forty-two. One first writes the two Arabic numerals in a particular array, one directly above the other, then, in a series of familiar steps performs the usual calculation. As one writes down the appropriate numerals in the appropriate places, one may well talk to oneself ('Let's see; two times seven is fourteen, so . . . '). Nevertheless, the calculation is not in the words one utters, if one does utter any words, even silently (and one need not), but in the written notation itself. Just imagine trying to do the calculation without appeal to Arabic numeration! One calculates *in* Arabic numeration. In much the same sense, I am suggesting, one reasons *in* a Euclidean diagram.

[24]That the words of the demonstration in Euclid are merely a record of the speech of someone presenting the demonstration to an audience is also indicated by the way the written text is presented in Euclid: "unspaced, unpunctuated, unparagraphed, aided by no symbolism related to layout." "Script," in Euclid, "must be transformed into pre-written language, and then be interpreted through the natural capacity for seeing [better: hearing] form in [spoken] language" (Netz [1999]: 163).

(This would perhaps be even more obvious if we considered someone trying to discover a demonstration of some proposition. The inscribing would not be of words but of diagrams.)

As this simple demonstration again shows, in order to follow, that is, to understand, a Euclidean demonstration, one must be able to see various drawn lines and points now as parts of one iconic figure and now as parts of another. The drawn lines are assigned very different significances at different stages in one's reasoning; they are *interpreted* differently depending on the context of lines they are taken, at a given stage in one's reasoning, to figure in. It is just this that explains the fruitfulness of a Euclidean demonstration, the fact that its conclusion is, as Kant would say, synthetic *a priori.* In Euclid, the desired conclusion is contained in the diagram, drawn according to one's starting point and the postulates, not merely implicitly, needing only to be made explicit (as in a deductive proof on the standard construal or in an Euler diagram), but instead only potentially. The potential of the diagram to demonstrate the conclusion is made actual only through an actual course of reasoning in the diagram, that is, by a series of successive refigurings of what it is that is being iconically represented by various parts of the diagram. Parts of wholes must be taken apart and combined with parts of other wholes to make quite new wholes. And this is possible, first, in virtue of the three levels of articulation in the diagram, and also because the various parts of the diagram signify geometrical objects only relative to ways of regarding those parts. A given line must actually be construed now as an iconic representation of (say) a part of another line and now as an iconic representation of a side of a square, if the demonstration is to succeed. The diagram, more exactly, its proper parts, must be actualized, now as this iconic representation and now as that, through one's construal of them as such representations, if the result is to emerge from what is given. Only a course of thinking through the diagram can actualize the truth that it potentially contains.[25]

Another example, this time of a reductio proof, is the demonstration of III.10, that a circle does not cut a circle at more points than two. The diagram (again lettered to enable us to explain in the written text what it involves) is shown in Figure 5.[26] The following is encoded, or

[25]We might, then, translate '*apodeixis*' not as 'proof' but as 'reasoning.'

[26]The image is taken from Heath [1956]: vol. 2, 23.

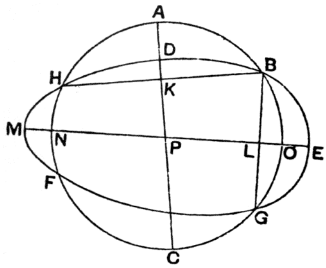
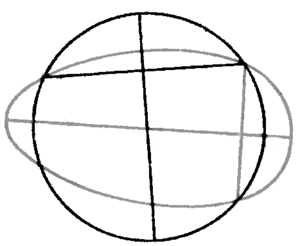
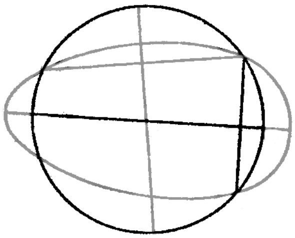


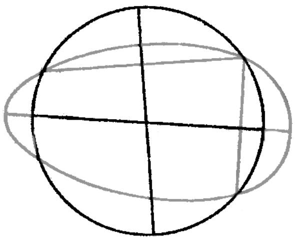
Figure 5

formulated, iconically in the diagram. First, we have by hypothesis (for reductio) that the circle ABC cuts the circle DEF at the points B, G, F, and H; and here (again) it is especially obvious that we do not *picture* the hypothesized situation, which is of course impossible, but instead formulate in the diagram the content of that hypothesis. BH and BG are drawn (licensed by the first postulate) and are then bisected at K and L respectively. (We know that we can do this from I.10.) KC is then drawn at right angles to BH, and LM at right angles to BG. (We know from I.11 that we can do this.) Both KC and LM are extended, KC to A and LM to E, as permitted by the second postulate. What we have then are, first, two straight lines BH and BG, both of which are chords to both circles; that is, we can see BH as drawn through ABC or as drawn through DEF, and similarly for BG. And we also know that AC bisects HB and is perpendicular to it, and similarly for ME and BG. Now the reasoning begins:

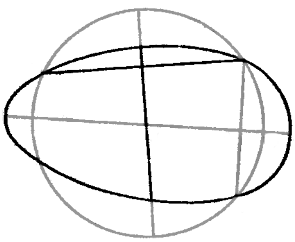
(1)  The center of the highlighted circle is on the highlighted bisecting line, by III.1 porism.

(2) 

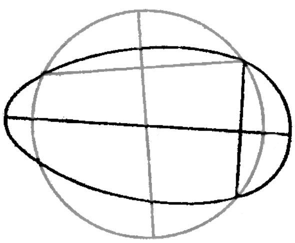
The center of the highlighted circle is on the highlighted bisecting line, again by III.1 porism.

(3) 

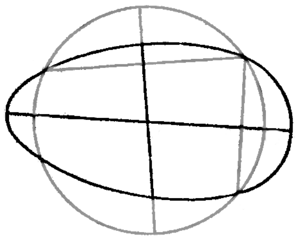
As the diagram shows, the two bisecting lines meet at only one point; so this point must be the center of the circle (because it is the only point that is on both lines).

(4) 

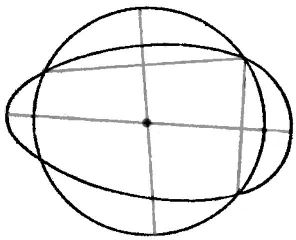
The center of the highlighted circle is on the highlighted bisecting line, by III.1 porism.

(5) 

Again, the center of the highlighted circle is on the highlighted bisecting line, by III.1 porism.

(6) 

The two bisecting lines meet at only one point and so that point must be the center of the circle (because, again, that is the only point on both lines known to contain the center).

(7) 

The two circles have the same center because one and the same point that was shown first (3) to be the center of the circle ABC was shown also (6) to be the center of DEF.

But we know from III.5 that if two circles cut one another then they will not have the same center. Our hypothesis that a circle cuts a circle at four points is false. Furthermore, because the demonstration involved only three of the four points presumed to cut the two circles, it clearly is cogent no matter how many points one supposes the two circles to cut at. Any number more than two will lead to contradiction. It follows that a circle does not cut a circle at more points than two.

In this demonstration we are concerned with lines, circles, more exactly, the circumferences of circles, and their interrelationships. All that we assume about the various figures is what we are told about them in the construction, and all that we infer at each stage is what we know follows from what we know about the figures that are taken at that stage to be iconically represented in the relevant part of the diagram. By focusing on the various parts of the diagram, now this part and now that, and conceptualizing these parts appropriately, then applying what one knows to the relevant figure thereby iconically represented, one can easily come to realize that, given what is known and assumed, the two circles have one and the same center. But this is absurd given that they cut one another. The assumption that a circle can cut a circle at more than two points must be rejected. It is in just this way that one reasons by reductio in a Euclidean diagram, by assuming in a diagram what one aims to show is false and then showing by a chain of strict and rigorous reasoning in the diagram that that assumption leads to a contradiction.

5 Conclusion

We have seen that diagrams in Euclid are not merely images or instances of geometrical figures but are instead icons with Gricean non-natural meaning. As such, they are inherently general. But as we also saw, this alone is not sufficient to explain the workings of a Euclidean demonstration. In particular, Euclidean reasoning cannot be merely diagram-based, even where the diagram is conceived iconically, because then no account can be given of the pop-up objects that are essential to the cogency of Euclidean demonstrations. Euclidean reasoning is instead properly diagrammatic; one reasons *in* the diagram in Euclidean geometry, actualizing at each stage some potential of the diagram. This is furthermore made possible, we have seen, by the nature and structure of the signs that make up that system. Because Euclidean diagrams are collections of primitive signs with three levels of articulation, the parts can be variously construed in systematic ways to mean, iconically represent, now this geometrical figure and now that. The conclusion that one draws on the basis of a Euclidean demonstration is, for just this reason, contained in one's starting points only potentially. The steps of the demonstration *must* be taken to actualize that conclusion. A Euclidean demonstration is not, then, diagram-based, its inferential steps licensed by various features of the diagram. It is properly diagrammatic. One reasons *in* the diagram, in Euclid, that is, through lines, *dia grammon*, just as the ancient Greeks claimed (Netz [1999]: 36).

The picture of Euclid's *Elements* that has emerged is, then, very different from that with which we began. First, a Euclidean demonstration, as we have come to understand it, is not a *proof* in the standard sense at all; it is not, that is, a sequence of sentences some of which are premises and the rest of which follow in a sequence of steps that are deductively valid, or diagram-based. Indeed, the demonstration does not lie in *sentences* at all. It lies in a certain activity, in a course of reasoning focused directly on a diagram. And this activity is needed, we have seen, because it alone can actualize the potential of the diagram to yield something new. The diagram, the actual marks on the page (or some appropriate analogue), must be seen now this way and now that if the result that is wanted is to be achieved. But if so, then Euclid's geometry is not an axiomatic system, that is, a collection of primitive and derived sentences; it does not involve reasoning about an instance (of a geometrical figure); and it is not merely diagram-based. It is instead a mode of mathematical inquiry, a mathematical practice that uses diagrams to explore the myriad discoverable necessary relationships that obtain among geometrical concepts, from the most obvious to the very subtle.

Acknowledgments

This paper is a much longer version of the paper I presented at the PMP 2007 conference on the philosophy of mathematical practice in Brussels. I have been helped in preparing this version by comments Ken Manders made at the time.

Bibliography

Azzouni, J. [2004]. "Proof and Ontology in Euclidean Mathematics." *New Trends in the History and Philosophy of Mathematics*. Ed. by T.H. Kjeldsen; S.A. Pederson; and L.M. Sonne-Hansen. Odense, Denmark: University Press of Southern Denmark.

Dipert, R.R. [1996]. "Reflections on Iconicity, Representation, and Resemblance: Pierce's Theory of Signs, Goodman on Resemblance, and Modern Philosophies of Language and Mind." *Synthese* **106**. 373–97.

Grice, H.P. [1957]. "Meaning." *Philosophical Review* **66**. 377–88.

Heath, T.L. [1956]. *The Thirteen Books of the Elements*. 2nd ed. First published in 1908. Toronto: Dover.

Lachterman, D.R. [1989]. *The Ethics of Geometry: A Genealogy of Modernity*. New York and London: Routledge.

Leibniz, G.W. [1969]. *Philosophical Papers and Letters*. Trans. by L.E. Loemker. Dordrecht: Reidel.

Macbeth, D. [2004]. "Viète, Descartes, and the Emergence of Modern Mathematics." *Graduate Faculty Philosophy Journal* **25**. 87–117.

— [2005]. *Frege's Logic*. Cambridge, MA: Harvard University Press.

Manders, K. [1996]. "Diagram Content and Representational Granularity." *Logic, Language, and Computation*. Ed. by J. Seligman and D. Westerståhl. Vol. I. Stanford: CSLI Publications and Stanford University Press.

— [2008]. "The Euclidean Diagram." *The Philosophy of Mathematical Practice*. Ed. by M. Paolo. Oxford: Oxford University Press.

Mueller, I. [1974]. "Greek mathematics and Greek logic." *Ancient Logic and its Modern Interpretation*. Ed. by J. Corcoran. Dordrecht and Boston: D. Reidel.

— [1981]. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. Cambridge, MA and London: MIT Press.

Netz, R. [1999]. *The Shaping of Deduction in Greek Mathematics*. Cambridge: Cambridge University Press.

Peirce, C.S. [1931]. *Collected Papers of Charles Sanders Peirce*. Ed. by C. Hartshorne and P. Weiss. Vol. 2. Cambridge, MA: Harvard University Press.

Russell, B. [1956]. "Mathematical Logic Based on the Theory of Types." *Logic and Knowledge: Essays 1901–50*. Ed. by R.C. Marsh. Book first published in 1956. Article first published in 1908. London and New York: Routledge.

Ryle, G. [1950]. "'If,' 'So,' and 'Because'." *Philosophical Analysis: A Collection of Essays*. Ed. by M. Black. Ithaca, NY: Cornell University Press.

Selfe, L. [1977]. *Nadia: A Case of Extraordinary Drawing Ability in Children*. London: Academic Press.

Shin, S.-J. [2002]. *The Iconic Logic of Peirce's Graphs*. Cambridge, MA and London: Bradford Books and MIT Press.

Stein, H. [1988]. "*Logos*, Logic, and *Logistiké*: Some Philosophical Remarks on the Nineteenth Century Transformation of Mathematics." *History and Philosophy of Modern Mathematics*. Ed. by W. Aspray and P. Kitcher. Vol. XI. Minnesota Studies in the Philosophy of Science. Minneapolis, MN: University of Minnesota Press.