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An Analysis of Poly-connexivity

Abstract. Francez has suggested that connexivity can be predicated of connectives other than the conditional, in particular conjunction and disjunction. Since connexivity is not any connection between antecedents and consequents—there might be other connections among them, such as relevance—, my question here is whether Francez's conjunction and disjunction can properly be called 'connexive'. I analyze three ways in which those connectives may somehow inherit connexivity from the conditional by standing in certain relations to it. I will show that Francez's connectives fail all these three ways, and that even other connectives obtained by following more closely Wansing's method to get a connexive conditional, fail to be connexive as well.

Keywords: Connexivity, Conjunction, Disjunction, Falsity condition, Residuation.

Introduction

Let for the moment $\sim$ and $\rightarrow$ be a generic negation and a generic conditional, respectively. A logic **L** is *connexive* only if

$\models_{\mathbf{L}} \sim(A \rightarrow \sim A)$	Aristotle's Thesis
$\models_{\mathbf{L}} \sim(\sim A \rightarrow A)$	Variant of Aristotle's Thesis
$\models_{\mathbf{L}} (A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$	Boethius' Thesis
$\models_{\mathbf{L}} (A \rightarrow \sim B) \rightarrow \sim(A \rightarrow B)$	Variant of Boethius' Thesis
$\not\models_{\mathbf{L}} (A \rightarrow B) \rightarrow (B \rightarrow A)$	Non-symmetry of Implication

Derivatively, a conditional is said to be connexive only if it appears as the conditional in the schemas above.[1]

Francez [6] has suggested that connexivity can be predicated of connectives other than the conditional, and proposes a logic, **PCON**, in which components of conjunctions and disjunctions exhibit certain connection. However, given that connexivity is not any connection between antecedents

[1] Or at least as the conditional in all antecedent and consequent parts, if something like Pizzi's *weak Boethius' Thesis*, $(A \rightarrow B) \supset \sim(A \rightarrow \sim B)$, is accepted as a connexive principle.

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and consequents—there might be other connections among them, such as relevance—, my question here is whether Francez's conjunction and disjunction can properly be called 'connexive'.

To answer that question, I analyze three ways in which those connectives may somehow inherit connexivity from the conditional by standing in certain relations to it. One of them is that their truth and falsity conditions are obtained by following the same method to obtain the truth and falsity conditions of a connexive conditional. The second one mimics the situation in relevance logic, where the corresponding intensional conjunction and disjunction are obtained through residuation relations. Finally, the conditional in the schemas above may be treated as a parameter that may be substituted by other binary connectives, and those which validate the schemas may deserve to be called 'connexive".

I will show that Francez's connectives fail all these three ways, but not only that: even other connectives obtained by following more closely Wansing's method to get a connexive conditional, fail the other two ways as well. These results strongly suggest that Francez's connectives are not really connexive, and cast serious doubts on the whole idea of poly-connexivity. In the process, the importance of giving sufficiently fine-grained descriptions of the modification in an evaluation condition will become evident.

The plan of the paper is as follows. In Section 1 I provide a quick review of **PCON** to make the text as self-contained as possible. There I fix my working logic, which is not **PCON** but an extension of it, with a tabular presentation which facilitates both discussion and calculations. Then, Sections 2–4 are each devoted to one of the ways that can make Francez's connectives connexive, and show that neither them nor other akin connectives achieve the connexive status. It is worth emphasizing that in doing so I use not only Francez's (alleged) connexive conditional, but also Wansing's more standard one. In Section 5 I revisit Wansing's method. I show that even the connectives obtained through the amended understanding of Wansing's method fail residuation and the validation of schemas. Then I probe the method to obtain connexive connectives by modifying falsity conditions not *à la* Wansing, but modifying falsity conditions *à la* Francez, because he modifies the falsity condition for the conditional in a way too distinct from Wansing's. The results are largely discouraging nonetheless. Finally, in the conclusions I offer recap and some prospects for rescuing poly-connexivity but which deserve separate treatment.

1. A Quick Review of PCON

Consider a language L consisting of formulas built, in the usual way, from propositional variables *Prop* with the connectives $\{\sim, \wedge_F, \vee_F, \rightarrow_F\}$. I will use the first capital letters of the Latin alphabet, 'A', 'B', 'C', ... as variables ranging over arbitrary formulas.

A model for **PCON** is a triple $\langle I, \leq, v\rangle$, where:

- I is a non-empty set of indexes of evaluation;
- $\leq$ is a partial order on I,
- $V : Prop \times I \longrightarrow \{\{\ \}, \{0\}, \{1\}, \{0, 1\}\}$ is an assignment of truth values to pairs of states and atomic formulas with the condition that $v_a \in V(p, i_1)$ and $i_1 \leq i_2$ only if $v_a \in (p, i_2)$ for all $p \in Prop$, $i_1, i_2 \in I$ and $v_a \in \{0, 1\}$.

Valuations V are then extended to interpretations σ to index-formula pairs by the following conditions:

$\sigma(p, i) = V(p, i)$
$1 \in \sigma(\sim A, i)$ iff $0 \in \sigma(A, i)$
$0 \in \sigma(\sim A, i)$ iff $1 \in \sigma(A, i)$
$1 \in \sigma(A \wedge_F B, i)$ iff $1 \in \sigma(A, i)$ and $1 \in \sigma(B, i)$
$0 \in \sigma(A \wedge_F B, i)$ iff either $1 \in \sigma(A, i)$ and $0 \in \sigma(B, i)$, or $0 \in \sigma(A, i)$ and $1 \in \sigma(B, i)$
$1 \in \sigma(A \vee_F B, i)$ iff $1 \in \sigma(A, i)$ or $1 \in \sigma(B, i)$
$0 \in \sigma(A \vee_F B, i)$ iff $0 \in \sigma(A, i)$ or $0 \in \sigma(B, i)$
$1 \in \sigma(A \rightarrow_F B, i)$ iff for all $x \in I$: $i \leq x$ and $1 \notin \sigma(A, x)$, or $1 \in \sigma(B, x)$
$0 \in \sigma(A \rightarrow_F B, i)$ iff either for all $x \in I$ such that $i \leq x$, $1 \notin \sigma(A, i)$ or $0 \in \sigma(B, i)$, or for all $x \in I$ such that $i \leq x$, $0 \notin \sigma(A, i)$ or $1 \in \sigma(B, i)$

A *biconditional*, $A \leftrightarrow_F B$, can be defined as $(A \rightarrow_F B) \wedge_F (B \rightarrow_F A)$.

Now, let Γ be a set of formulas of **PCON**. A is a *logical consequence* of Γ in **PCON**, $\Gamma \models_{\mathbf{PCON}} A$, if and only if, for every evaluation σ, $1 \in \sigma(A, i)$ if $1 \in \sigma(B, i)$ for every $B \in \Gamma$. A is *valid* or *holds* in **PCON** if and only if $\Gamma \models_{\mathbf{PCON}} A$ and $\Gamma = \varnothing$.[2]

[2]In fact, these definitions will be used for any other logic employed below, not only **PCON**.

PCON has the following remarkable properties:

- It does not validate the De Morgan principles, neither in arrow nor in rule form. For example, consider the case where $\sigma(A, i) = \{1, 0\}$ and $\sigma(B, i) = \{\ \}$. Then $1 \in \sigma({\sim} A \vee_F {\sim} B, i)$ yet $1 \notin \sigma(A \wedge_F B, i)$.
- It is contradictory, or $\sim$-inconsistent. It validates, among others, the following pairs of schemas:

  $(A \wedge_F {\sim} A) \rightarrow_F A$; ${\sim}((A \wedge_F {\sim} A) \rightarrow_F A)$

  $A \rightarrow_F (A \vee_F {\sim} A)$; ${\sim}(A \rightarrow_F (A \vee_F {\sim} A))$

For simplicity, let me consider the extension of **PCON** where W is a singleton.[3] This logic, which I call '**PCON***', admits the following truth-tabular presentation:

$\sim A$	A
$\{0\}$	$\{1\}$
$\{1,0\}$	$\{1,0\}$
$\{\ \}$	$\{\ \}$
$\{1\}$	$\{0\}$

$A \wedge_F B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{\ \}$	$\{\ \}$

$A \vee_F B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{\ \}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$

[3] The logic has more validities and less invalidities than **PCON**, which is a good thing when one aims at validating certain schemas. Additionally, this will not cause trouble in what follows when it comes to the invalidation of Symmetry, $\models_{\mathbf{L}} (A \rightarrow B) \rightarrow (B \rightarrow A)$.

$A \to_F B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

As usual, '$\{1\}$' stands for *is assigned truth only*, '$\{0\}$' stands for *is assigned falsity only*, '$\{1,0\}$' stands for *is assigned both truth and falsity* and '$\{\ \}$' stands for *is assigned neither truth nor falsity.*

With the exception of negation, the more common connectives, the *extensional* ones, are not available in **PCON***. It is useful to make their evaluation conditions explicit at this point to facilitate comparison in the next sections.

$1 \in \sigma(A \wedge B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$
$0 \in \sigma(A \wedge B)$ iff either $0 \in \sigma(A)$ or $0 \in \sigma(B)$

$1 \in \sigma(A \vee B)$ iff either $1 \in \sigma(A)$ or $1 \in \sigma(B)$
$0 \in \sigma(A \vee B)$ iff $0 \in \sigma(A)$ and $0 \in \sigma(B)$

The tabular presentation of the above conditions is as follows:

$A \wedge B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

$A \vee B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{\ \}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$

An *extensional* conditional $A \to B$ is definable with the above connectives as $\sim A \vee B$, i.e., it is true iff the antecedent is false or the consequent is true:

$1 \in \sigma(A \to B)$ iff $0 \in \sigma(A)$ or $1 \in \sigma(B)$
$0 \in \sigma(A \to B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

With the corresponding truth table:
An *extensional biconditional*, $A \leftrightarrow B$, can be defined as $(A \to B) \wedge (B \to A)$.

$A \to B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{\ \}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

A *material* conditional, $A \supset B$, evaluated as follows, would be useful as well:

$1 \in \sigma(A \supset B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$
$0 \in \sigma(A \supset B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$
Its truth table is as follows:

$A \supset B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

A *material biconditional*, $A \equiv B$, can be defined unsurprisingly as $(A \supset B) \wedge (B \supset A)$.

In many cases below, I will consider other conjunctions and disjunctions accompanying $\to_F$ instead of $\wedge_F$ and $\vee_F$; in that case, the set of connectives will be $\{\sim, \wedge_X, \vee_Y, \to_F\}$—where '$X$' and '$Y$' stand for suitable subscripts—and the language will change accordingly. In other cases, another conditional will accompany $\wedge_F$ and $\vee_F$; in that case, the set of connectives will be $\{\sim, \wedge_F, \vee_F, \to_Z\}$—where '$Z$' stands again for a suitable subscript—and the language will change accordingly again. Finally, there will be cases in which the whole set of binary connectives changes, with the corresponding change in language. Changes in language will be left implicit and will be indicated by the *de facto* use of different connectives.

2. Poly-connexivity Through Wansing's Method

Wansing [21] obtained the connexive logic **C** by modifying the falsity condition for the conditional in Nelson's logic **N4**. Since then, several logics have been obtained by following both Nelson's example, starting in [10], of giving falsity a separate treatment from truth, and Wansing's example of changing the falsity condition of the conditional.[4] Wansing modified the evaluation conditions of the material conditional

$1 \in \sigma(A \supset B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \supset B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

into the following ones[5]:

$1 \in \sigma(A \rightarrow_W B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \rightarrow_W B)$ iff $1 \notin \sigma(A)$ or $0 \in \sigma(B)$

that is, he left the truth condition untouched and modified the falsity condition. (The exact kind of modification done will be discussed later.)

Francez describes how he develops his logic thus:

> I apply the proposed methodology [changing the falsity condition of the conditional] to other connectives, conjunction and disjunction too, endowing them too with the interaction with negation different than the classical (boolean) one. Such an interaction also captures connections between the arguments of the other binary connectives, transcending their truth. By this move, I expand the scope of connexivity from the conditional only to a more extensive signature of connectives. Therefore, I coin the resulting logic *PCON* —a poly-connexive logic.

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[4] Omori [12] used the same ideas on top of **LFI1** to get another connexive logic, **dLP**. After that, he has shown (cf. [11]) that a number of well-known and new paraconsistent and relevant logics can be obtained also by changing appropriately the falsity condition for some connectives while leaving the **FDE**-like truth and falsity conditions for the remaining ones, in most cases even negation, fixed. More recently, Wansing and Unterhuber [25] modified the falsity condition of Chellas' basic conditional logic **CK** and they obtained a (weakly) connexive logic. Omori and Wansing [14] have generalized that idea of changing a falsity condition to other connectives to get other contra-classical logics, not only connexive logics in the ballpark of Wansing's **C**. Even more recently, Omori and Wansing [15] have put forward a systematization of connexive logics based on certain controlled tweakings in the conditional's truth and falsity conditions.

Such a general approach to non-classical logic—roughly, start with **FDE**-like truth and falsity conditions and modifying any of them to obtain semantics for all sorts of logics—deserves a name. I use 'the Bochum Plan', in analogy with the Australian, American and Scottish plans for relevance, although the scope of the Bochum Plan is far broader than paraconsistency and relevance, as the contra-classical logics obtained show.

[5] The logic resulting from adding Wansing's conditional to **FDE** is called '**MC**' and it has been studied in [22].

This is a non-sequitur. Even if changing the falsity condition of a formula $A©B$ could ensure a (meaning) connection between its components—which remains to be shown—, it is not clear that such connection is a connexive one: there are many (intensional) connections between antecedent and consequent in a valid conditional, and the claim that some other connections that are not necessarily about conditionals count as connexive needs argument. Likewise, without further argument, the mere connection between conjuncts and disjuncts is not enough to qualify conjunctions and disjunctions as connexive.[6]

There is a sense in which Francez is indeed applying Wansing's method to other connectives, namely Francez is modifying the falsity condition for the conditional, conjunction and disjunction, just as Wansing modified the falsity condition of the conditional. Nonetheless, there is another sense in which Francez is not applying Wansing's method, and it has to do with the fine-grainedness in the description of Wansing's method. Consider someone that modified the truth condition of conjunction. Such a person could say that they are following Wansing's method, because the method consists in modifying the evaluation condition of connectives. In a sense, such a person would be right. However, one could say that in a more important respect, they are wrong, since Wansing is not modifying any evaluation condition, but the falsity condition.

One could object Francez's procedure along similar lines. In a sense, it is true that Wansing modifies the falsity condition, but he seems to do more than that. The usual evaluation conditions for conjunction, disjunction, conditional (and biconditional) have the following general shape

$1 \in \sigma(A©B)$ iff $1 \in \sigma(A)$ *copyright* $1 \in \sigma(B)$

$0 \in \sigma(A©B)$ iff $v_i \in \sigma(A)$ *connective* $0 \in \sigma(B)$

where $v_i \in \{1, 0\}$, '*copyright*' stands for a metalinguistic counterpart of © and '*connective*' stands for a metalinguistic counterpart of some other connective, in general distinct from ©.[7] For simplicity, let me assume that

[6] Richard Sylvan—see [20], [?]—used the term 'sociative' to name all those logics in which antecedent and consequent in a valid conditional must be definitely connected in addition to exhibit certain combinations of truth values. Among such definite connections there is relevance, connexivity, meaning containment, etc. It seems to me that Francez wants to use 'connexivity' for sociativity.

[7] Introducing other binary connectives, such as dual implication, would require a more abstract version of the evaluation conditions since in the truth condition appears more than one value. Nonetheless, that situation can be safely ignored in this discussion.

classical logic applies to the metalinguistic connectives, so that for example "If A is true then B is true" is equivalent to "A is not true or B is true".

Consider now Wansing's modification of the evaluation conditions, from

$1 \in \sigma(A \supset B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \supset B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

to

$1 \in \sigma(A \rightarrow_W B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \rightarrow_W B)$ iff $1 \notin \sigma(A)$ or $0 \in \sigma(B)$

This is a very specific modification: it seems to consist in putting the same metalinguistic connective in both the truth and falsity conditions, a material conditional, and Francez does not do that.

In applying Wansing's method to conjunction and disjunction, i.e., putting the same metalinguistic connective in both their truth and falsity conditions, one obtains Francez's disjunction indeed, but not his conjunction. The truth and falsity conditions for conjunction would be the following ones:

$1 \in \sigma(A \wedge_I B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$

$0 \in \sigma(A \wedge_I B)$ iff $0 \in \sigma(A)$ and $0 \in \sigma(B)$

$A \wedge_I B$	$\{1\}$	$\{1, 0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{1, 0\}$	$\{1\}$	$\{1, 0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$

Both connectives are old acquaintances: they are Arieli and Avron's informational meet and informational join, respectively, of their logic $\mathbf{BL}_{\supset}$ in [2]. (That is why I write such conjunction as '$A \wedge_I B$'.)

One question at this point is whether applying Wansing's method as described here is enough to ensure connexivity for the connectives it is applied to. I will give a negative answer to this question in the next section, where Francez's connectives are given another try. (And other possible renderings of Wansing's method are discussed in Section 5.)

3. Poly-connexivity Through Residuation

Francez could reply that even if he did not follow Wansing's method to the letter, if his conditional $A \rightarrow_F B$ is connexive, $A \wedge_F B$ and $A \vee_F B$ are connexive as well. The reasoning could be as follows. Just as a *fusion* $A \circ B$ is the relevant intensional conjunction associated to a relevant implication $A \rightarrow B$ because the former is the (left) residual of the latter in a suitable relevance logic **L**, i.e.,

$$A \circ B \models_{\mathbf{L}} C \text{ iff } A \models_{\mathbf{L}} B \rightarrow C$$

$A \wedge_F B$ is the (left) residual of $B \rightarrow_F C$, so $A \wedge_F B$ is as connexive as $A \rightarrow_F B$ because the following holds in $\mathbf{PCON}^*$:

$$A \wedge_F B \models_{\mathbf{PCON^*}} C \text{ iff } A \models_{\mathbf{PCON^*}} B \rightarrow_F C$$

(In fact, this holds already in **PCON**.)

This approach to poly-connexivity through residuation deserves some comments. First, consider a logic **L** based on a language which contains all of $\sim$, $\wedge$, $\wedge_F$, $\rightarrow_F$, $\rightarrow_W$ and $\supset$. Under the above understanding of "connexive conjunction", extensional conjunction $A \wedge B$ is a left residual in **L** of both $A \rightarrow_F B$ and Wansing's $A \rightarrow_W B$ as well. Thus, in **L**, $A \wedge B$ would be as connexive as $A \rightarrow_F B$ and $A \rightarrow_W B$. But then no modification in the falsity conditions is needed to get such a connexive conjunction. The problem is worse since the extensional conjunction is a (left) residual of the material conditional. This leads to the unwelcome result that, in **L**, the material conditional is as connexive as Francez's (or Wansing's) conditional![8]

Moreover, since the De Morgan laws are not valid in $\mathbf{PCON}^*$, the alleged connexivity of $A \vee_F B$ is not immediate. Suppose that if $A \rightarrow_F B$ is a connexive connective and so it is its (left) residual then, if the coimplication associated to $A \rightarrow_F B$ is a connexive connective, so it is its (right) residual. The coimplication $A \leftarrow B$ associated to a conditional $A \rightarrow B$ is obtained by exchanging the truth and falsity conditions of the latter. Thus, the evaluation conditions for $A \leftarrow_F B$ are as follows:

$1 \in \sigma(A \leftarrow_F B)$ iff $1 \notin \sigma(A)$ or $0 \in \sigma(B)$, or $0 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \leftarrow_F B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

And this is its truth table:

[8] $\wedge_I$ would face exactly the same problem were it proposed as a connexive conjunction.

$A \leftarrow_F B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{\ \}$	$\{1\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

Hence, in many cases, a coimplication can be defined as a negation of a conditional; $A \leftarrow_F B$ can be so defined, as $\sim(A \rightarrow_F B)$, in $\mathbf{PCON}^*$.

$A \vee_F B$ is not the (right) residual of $A \leftarrow_F B$, that is,

$$C \models_{\mathbf{PCON}^*} A \vee_F B \quad \text{iff} \quad C \leftarrow_F A \models_{\mathbf{PCON}^*} B$$

does not hold. For a countermodel, put A instead of C. Then $A \models_{\mathbf{PCON}^*} A \vee_F B$ is valid but $A \leftarrow_F A \models_{\mathbf{PCON}^*} B$ is not; as a witness, take a σ where $\sigma(A) = \sigma(B) = \{\ \}$. (In fact, this fails already in **PCON** and the same countermodels apply when residuation is evaluated not with $A \leftarrow_F B$ but with $A \leftarrow_W B$, in a suitable logic containing both $\vee_F$ and $\leftarrow_W$.)

This also shows that the application of Wansing's method does not ensure connexivity for the connectives it is applied to: informational disjunction (i.e., Francez's disjunction) does not residuate coimplications associated to connexive conditionals, contrary to what it is expected, and informational conjunction residuates both Francez's and Wansing's conditional, but it would be as connexive as the extensional conjunction.

One might wonder whether the coimplications above are connexive connectives themselves. The appeal to coimplications was trying to be helpful to Francez view in the following way. Suppose that the coimplication associated to a connexive conditional has some degree of connexivity, no matter how and why. But even assuming this, such alleged connexivity does not go to Francez's disjunction, or at least it cannot be detected through residuation. If it is not connexive to begin with, that would be worse for the attempt to defend the connexivity of Francez disjunction.

In many contexts where one treats truth and falsity independently, intersubstitutivity is not going to obtain unless two formulas are equivalent and also their negations are equivalent. Maybe what is necessary here is to take into account the negative half of the notion of residuation that could help to distinguish between Francez's conjunction from the extensional one, namely

$$\sim(A \wedge_F B) \models_{\mathbf{L}} \sim C \quad \text{iff} \quad \sim A \models_{\mathbf{L}} \sim(B \rightarrow_F C)$$

Negative residuation is a most interesting idea,[9] but it is not going to do the job, either. Consider $\sim A \models_{\mathbf{PCON*}} \sim (B \rightarrow_F B)$: it is valid, but $\sim (A \wedge_F B) \models_{\mathbf{PCON*}} \sim B$ is not. (As a countermodel, consider that B is just true and A is both true and false.)

So far the enterprise seems hopeless, but perhaps this is only an impression originated by focusing on the wrong conditionals. Wansing's and Francez's conditionals are controversial even by connexive lights. Wansing's conditional is *hyper-connexive*, that is, it validates the converses of Boethius' Theses, and there is a debate on their plausibility. (See [9,24].) On the other hand, Francez's conditional validates both $(\sim A \rightarrow_F B) \rightarrow_F \sim (A \rightarrow_F B)$ and $(A \rightarrow_F B) \rightarrow_F \sim (\sim A \rightarrow_F B)$, and they have also been questioned. (See [13,23].) One might wonder[10] whether a different result might be obtained in detaching Francez's conjunction and disjunction from his own conditional but also from Wansing's, while coupling them to other connexive conditionals that do not validate the above controversial theses.

One of them is Angell–McCall's conditional in the logic **CC1** (cf. [1,8]), defined as follows:
$1 \in \sigma(A \rightarrow_{AM} B)$ iff $0 \in \sigma(A)$ iff $0 \in \sigma(B)$, and $1 \notin \sigma(A)$ or $1 \in \sigma(B)$
$0 \in \sigma(A \rightarrow_{AM} B)$ iff $0 \in \sigma(A)$ and $0 \notin \sigma(B)$, or $0 \in \sigma(B)$ and $0 \notin \sigma(A)$

$A \rightarrow_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{\ \}$
$\{\ \}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$

Unfortunately for Francez's conjunction, it is not even the residual of $\rightarrow_{AM}$ when they are together in a logic **L**. Consider $A \wedge_F B \models_{\mathbf{L}} A$. While it is logically valid, $A \models_{\mathbf{L}} B \rightarrow_{AM} A$ is not. For a countermodel, suppose that both A and B are true, but B is also false and A is not. In that case $B \rightarrow_{AM} A$ is just false.[11]

Evaluating residuation for Francez's disjunction using the coimplication associated to the Angell-McCall conditional introduces complications, already in the definition of the coimplication. The Angell-McCall conditional

[9] And I thank a reviewer for suggesting me to take it into account.

[10] As a referee did, and I thank them for urging me to discuss explicitly this issue.

[11] The same model rules $\wedge_I$ out. With **CC1** on scene, one might wonder whether its conjunction could play the required role of a connexive conjunction. It is defined as follows:
$1 \in \sigma(A \wedge_{AM} B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$

is connexive in the presence of Boolean negation, not the de Morgan one. But using Boolean negation and defining then $A \leftarrow_{AM} B$ as $\neg(A \rightarrow_{AM} B)$ allows easy counterexamples to residuation. For example, consider a logic $\mathbf{L}$ that includes Boolean negation, the Angell-McCall conditional and Francez's disjunction. Then $C \models_{\mathbf{L}} A \vee_F \neg A$ would be valid, but $C \leftarrow_{AM} A \models_{\mathbf{L}} \neg A$ would not. (For a countermodel, consider let C be both true and false, and A just true.) The options are either using de Morgan negation, which would need very strong motivation since it is not a good match for $\rightarrow_{AM}$, or allowing mixtures of both negations. Without a principled guide on what combinations to allow to avoid counterexamples, I leave the matter here.

Therefore, the following claims are incompatible:

- residuation delivers poly-connexivity;
- Francez's or the informational connectives are binary connexive connectives in addition to the conditional;
- the coimplications associated to the various connexive conditionals are those that were given here.

The approach to coimplication is the standard in the literature, so rather there are no binary connexive connectives other than the conditional, or they are not Francez's, or the approach to poly-connexivity through residuation is not right. Since residuation gives the result that extensional conjunction is a residual of both Francez's and Wansing's conditional, it seems that, in spite of its success in the realm of relevance, residuation is not a good guide to find the intensional connectives associated to connexive arrows. (Which, as far as we know, is still compatible with the both non-existence of connexive connectives besides the conditional or the non-uniqueness of connexive conjunction and disjunction.)

$0 \in \sigma(A \wedge_{AM} B)$ iff $0 \in \sigma(A)$ and $0 \notin \sigma(B)$, or $0 \in \sigma(B)$ and $0 \notin \sigma(A)$

$A \wedge_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{0\}$	$\{\}$
$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$

It is not a left residual of $\rightarrow_{AM}$, the same countermodel as with $A \wedge_F B$ applies here, and although it is a left residual of Francez's and Wansing's conditionals, we are back into the problem of extensional conjunction being also a right residual of those connectives.

4. Poly-connexivity Through Schemas

Let me consider now an approach to poly-connexivity through schemas. Suppose that the binary connective in the connexive schemas is treated as a parameter, in a way that the connexivity of a binary connective ⓒ entails (in a logic **L**) the validity of the connexive schemas substituting uniformly the arrows by ⓒ as follows:

$\models_{\mathbf{L}} \sim(A ⓒ \sim A)$ ⓒ-Aristotle's Thesis

$\models_{\mathbf{L}} \sim(\sim A ⓒ A)$ Variant of ⓒ-Aristotle's Thesis

$\models_{\mathbf{L}} (A ⓒ B) ⓒ \sim(A ⓒ \sim B)$ ⓒ-Boethius' Thesis

$\models_{\mathbf{L}} (A ⓒ \sim B) ⓒ \sim(A ⓒ B)$ Variant of ⓒ-Boethius' Thesis

$\not\models_{\mathbf{L}} (A ⓒ B) ⓒ (B ⓒ A)$ ⓒ-Non-symmetry of ⓒ

However, neither $A \wedge_F B$ nor $A \vee_F B$ are connexive in this sense, either. It can be easily verified that

$\not\models_{\mathbf{PCON}^*} (A ⓒ B) ⓒ \sim(A ⓒ \sim B)$

$\not\models_{\mathbf{PCON}^*} (A ⓒ \sim B) ⓒ \sim(A ⓒ B)$

for ⓒ $\in \{\wedge_F, \vee_F\}$: simply consider the case where both A and B are neither true nor false. Such assignment also shows that using informational conjunction uniformly in the schemas above does not do the job.

Another option is that when the main connective is binary, it is left as a conditional, obtaining thus what I call 'ⓒ $\rightarrow$ thesis'. Abusing notation a little bit here, and using momentarily '$\wedge$' and '$\rightarrow$' as a generic conjunction and a generic conditional, respectively, one should get the following schemas for conjunction:

$\models_{\mathbf{L}} \sim(A \wedge \sim A)$ $\wedge\ \rightarrow$-Aristotle's Thesis

$\models_{\mathbf{L}} \sim(\sim A \wedge A)$ Variant of $\wedge\ \rightarrow$-Aristotle's Thesis

$\models_{\mathbf{L}} (A \wedge B) \rightarrow \sim(A \wedge \sim B)$ $\wedge\ \rightarrow$-Boethius' Thesis

$\models_{\mathbf{L}} (A \wedge \sim B) \rightarrow \sim(A \wedge B)$ Variant of $\wedge\ \rightarrow$-Boethius' Thesis

$\not\models_{\mathbf{L}} (A \wedge B) \rightarrow (B \wedge A)$ Non-symmetry of $\wedge$

And the following ones for disjunction:

$\models_{\mathbf{L}} \sim(A \vee \sim A)$ $\vee\ \rightarrow$-Aristotle's Thesis

$\models_{\mathbf{L}} \sim(\sim A \vee A)$ Variant of $\vee\ \rightarrow$-Aristotle's Thesis

$\models_{\mathbf{L}} (A \vee B) \rightarrow \sim(A \vee \sim B)$ $\vee\ \rightarrow$-Boethius' Thesis

$\models_{\mathbf{L}} (A\vee \sim B) \rightarrow\sim(A \vee B)$ Variant of $\vee \rightarrow$-Boethius' Thesis

$\not\models_{\mathbf{L}} (A \vee B) \rightarrow (B \vee A)$ Non-symmetry of $\vee$

One general remark concerning the schemas for conjunction is worth noticing here: They are valid in classical logic, with the only exception of Non-symmetry. Given the machinery deployed in this paper, if a connexive conjunction is to be distinct from an extensional conjunction, it is because it meets Non-symmetry, or else one can distinguish them through Residuation (with only one of them validating it), modulo some connexive conditional.

In $\mathbf{PCON}^*$, the $\wedge_F\rightarrow_F$-Boethius' Theses are valid, but neither $\sim(A\wedge_F\sim A)$ nor its variant are valid —consider the case when A is neither true nor false—; moreover, Symmetry of $\wedge_F$ is valid. Thus, Francez's conjunction is not connexive in this sense, either.

Remember that $A \vee_F B$ is both Francez's proposed connexive disjunction and that it can be seen as the result of applying Wansing's method, so its alleged connexivity is supported in two fronts. However, neither the $\vee_F$-Aristotle's Theses nor the $\vee_F \rightarrow_F$-Boethius' Theses are valid—for the former, consider the case when A is neither true nor false; for the latter, consider the case when A is just true but B is not true—and the Symmetry of $\vee_F$ is valid again. Thus, Francez's (informational) disjunction is not connexive in this sense, either.

The situation is worse than the mere invalidity of the $\vee_F \rightarrow_F$-Boethius' Theses, though. Since $A\vee_F B$ is just true when A is just true and B is neither true nor false, but $\sim (A\vee_F \sim B)$ is just false in that case, there seems to be no suitable conditional that could recover the $\vee_F$-Boethius' Theses. This result seems most damaging for the prospects of poly-connexivity or, at any rate, of predicating connexivity from this disjunction.[12]

What about informational conjunction? It does not fare better than disjunction. When A is just true and B is at least true, $A \wedge_I B$ is just true, but $\sim(A\wedge_I \sim B)$ is untrue. In fact, $\sim(A\wedge_I \sim B)$ it is just false when A is just true and B is both true and false. In this case, most conditionals will fail to validate $(A \wedge_I B) \rightarrow\sim(A\wedge_I \sim B)$.

[12]There is yet another option allowed by combinatorics, namely that the main connective in the schemas is completely different from those in antecedent and consequent. Without any foreseeable argument to consider it a genuine option related whether to connexivity or to weak connexivity, I leave it aside.

5. Poly-connexivity Through Wansing's Method, Again, and Francez's Condition on Its Own

Remember that Wansing modified the (material) conditional

$1 \in \sigma(A \supset B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \supset B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

to the following one:

$1 \in \sigma(A \rightarrow_W B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \rightarrow_W B)$ iff $1 \notin \sigma(A)$ or $0 \in \sigma(B)$

I described this modification as putting the same metalinguistic connective in both the truth and falsity conditions. Nonetheless, one could describe it in a different way: Wansing put a metalinguistic (material) conditional in the falsity condition, and it was merely a coincidence that it was the same metalinguistic connective as that in the truth condition for the conditional.

Described in this way, the application of Wansing's method would lead to the following truth and falsity conditions for conjunction and disjunction:

$1 \in \sigma(A \wedge_c B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$

$0 \in \sigma(A \wedge_c B)$ iff $0 \notin \sigma(A)$ or $0 \in \sigma(B)$

$A \wedge_c B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$

$1 \in \sigma(A \vee_c B)$ iff $1 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \vee_c B)$ iff $0 \notin \sigma(A)$ or $0 \in \sigma(B)$

$A \vee_c B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1.0\}$	$\{1,0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$

This is of no help, though. The new disjunction is not a (right) residual of $A \leftarrow_F B$ nor of $A \leftarrow_W B$. (The same countermodels as in Section 3 apply here.) Now, consider the case where A is both true and false but B is neither true nor false. In that case, $(A \vee_c B)$ is just true but $\sim(A \vee_c \sim B)$ is just false, which again dooms most acceptable conditionals.[13]

Among the non-conditional binary connectives studied here, $A \wedge_c B$ is the closest to a connexive connective. This new conjunction is again a (left) residual of $A \rightarrow_F B$, but it also satisfies the $\wedge_c \rightarrow_F$ Boethius' Theses, although $(A \wedge_c B) \rightarrow_F (B \wedge_c A)$ gets validated. However, in all fairness, this is might not be bad result for connectives like conjunctions and disjunctions, even connexive ones. Moreover, $(A \wedge_c B) \rightarrow_F (B \wedge_c A)$ holds, but there are σ's such that $\sigma(A \wedge_c B) \neq \sigma(B \wedge_c A)$.

These nice features of $\wedge_c$ remain when $\rightarrow_W$ is used instead of $\rightarrow_F$. When one uses $\rightarrow_{AM}$, (left) residuation is lost, though. However, it is remarkable that in such case all the $\wedge_c \rightarrow_{AM}$ schemas are validated, including the non-symmetry of conjunction. Again, this is the closest to a non-conditional connexive so far given the criteria studied here.[14]

What if, instead of taking Wansing's method to obtain "connexive" connectives, one starts directly with Francez's strategy to obtain his connexive conditional? Remember that Francez changed the (material) conditional

$1 \in \sigma(A \supset B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \supset B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

to the following one:

$1 \in \sigma(A \rightarrow_F B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \rightarrow_F B)$ iff whether $1 \notin \sigma(A)$ or $0 \in \sigma(B)$, or $0 \notin \sigma(A)$ or $1 \in \sigma(B)$

Seen abstractly, by following Francez's method one turns the evaluation conditions of the form

$1 \in \sigma(A © B)$ iff $1 \in \sigma(A)$ *copyright* $1 \in \sigma(B)$

$0 \in \sigma(A © B)$ iff $v_i \in \sigma(A)$ *connective* $0 \in \sigma(B)$

—where again $v_i \in \{1, 0\}$—into the following ones:

[13] The verification that the schemas are not validated when these connectives substitute © uniformly is left to the reader.

[14] The verification that $\wedge_c$ does not validate the parameterized schemas putting it uniformly for each © is left once more to the reader. Some reader might be worried that $\wedge_c$ is then too conditional-ish and ceases to be a conjunction. But consider a logic **L** based on language where the only binary connective is $\wedge_c$. Then $\wedge_c$ delivers schemas expected from a conjunction and not from a conditional, as $A \wedge_c B \models_{\mathbf{L}} A$—and does not validate schemas that are typical from connexive conditionals in the **FDE**-vicinity, such as $A \wedge_c (B \wedge_c A)$.

$1 \in \sigma(A \copyright_* B)$ iff $1 \in \sigma(A)$ *copyright* $1 \in \sigma(B)$

$0 \in \sigma(A \copyright_* B)$ iff $1 \in \sigma(A)$ *copyright* $0 \in \sigma(B)$, or $0 \in \sigma(A)$ *copyright* $1 \in \sigma(B)$

Actually, $A \wedge_F B$ is already an instance of this procedure, but $A \vee_F B$ is not. The resulting connective, $A \vee_N B$ has the following evaluation conditions:

$1 \in \sigma(A \vee_N B)$ iff $1 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \vee_N B)$ iff whether $1 \in \sigma(A)$ or $0 \in \sigma(B)$, or $0 \in \sigma(A)$ or $1 \in \sigma(B)$

and the corresponding truth table:

$A \vee_N B$	$\{1\}$	$\{1, 0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1, 0\}$	$\{1, 0\}$	$\{1.0\}$	$\{1, 0\}$
$\{1, 0\}$	$\{1, 0\}$	$\{1, 0\}$	$\{1, 0\}$	$\{1, 0\}$
$\{\ \}$	$\{1, 0\}$	$\{1, 0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{1, 0\}$	$\{1, 0\}$	$\{0\}$	$\{0\}$

The reader can easily verify that the same countermodels to residuation (with regards to $A \leftarrow_W B$ and $A \leftarrow_F B$) apply here. And although this disjunction validates both $(A \vee_N B) \rightarrow_F \sim (A \vee_N \sim B)$ and $(A \vee_N \sim B) \rightarrow_F \sim (A \vee_N B)$, it fails $\sim (A \vee_N \sim A)$, its variant, and the invalidity of $(A \vee_N B) \rightarrow_F (B \vee_N A)$. Thus, there is no poly-connexivity in sight yet.[15]

Conclusions

In this paper I have attempted to make sense of the notion of poly-connexivity, that is, the idea that there are other connectives besides the conditional that can be considered connexive. I examined in particular Francez's proposed connexive conjunction and disjunctions. I used three approaches for trying to make sense of poly-connexivity: adapting Wansing's method to make the conditional connexive to the cases of conjunction and disjunction; through residuation *à la* relevance logic, and by parameterizing the connexive schemas to allow other binary connectives in them, not only the conditional. In the process, it became evident that what Wansing did to obtain a

[15]It was already shown in Section 4 that Francez's conjunction does not validate the parameterized schemas when uniformly put instead of ©. The verification that $A \vee_N B$ does not validate them either is left to the reader.

connexive logic can be described in different, non-equivalent ways, and that paying attention to sufficiently fine-grained descriptions of the modification in an evaluation condition is very important.

I showed that Francez's method to obtain his alleged connexive connectives is not exactly Wansing's. Then, it is not clear that Francez's strategy endows conjunction and disjunction with connexive properties. But even applying Wansing's method in what seems closer ways, it is doubtful that the connectives so obtained are connexive. The alleged connexive conjunctions do residuate connexive conditionals, but extensional conjunction does so as well. Moreover, the alleged connexive disjunctions do not residuate neither Wansing's nor Francez's coimplications. Finally, even if connexive schemas are parameterized and other binary connectives that validate them are going to be considered as connexive, none of the alleged connexive conjunctions and disjunctions considered here can do the job, with the possible exception of $\wedge_c$. These results are summarized in Table 1, with the question mark indicating that a case can be made for the connexivity of $\wedge_c$, as showed in Section 5. All this strongly suggests that Francez's connectives cannot be considered connexive, and casts serious doubts on the whole idea of poly-connexivity.[16]

Nonetheless, poly-connexivity might be achieved in more sophisticated terms. Here are some non-exclusive ideas left for further work:

Weak poly-connexivity The most common way to understand the connexive schemas is as schemas where all the occurrences of the connectives are of the same kind, that is, all the negations are the same and all the conditionals are the same. In [16], Pizzi characterized a *weak system of connexive logic* as a logic that validates Boethius' thesis in the form $(A \to B) \supset \sim(A \to \sim B)$, also known as *weak Boethius' thesis* in the context of consequential implication, see [17].

Then, an option for poly-connexivity is going weak, that is, distinguishing different kinds of conjunctions (respectively, disjunctions), and perhaps also negations, in the parameterized schemas of Section 4.

Relating poly-connexivity Jarmużek and Malinowski [7] have developed connexive logics in the framework of relating logic. These connexive logics extend the language of classical (zero-order) logic using conjunction, disjunction and Boolean negation by a "relating implication", $A \to^W B$, the

[16] This is not to say that Francez's connectives are useless or not worth investigating. In fact, they seem to be good tools to give a logical treatment of intonational stress; see [4].

Table 1. Properties of “connexive” connectives

Connectives	Obtained by	Residuation $\rightarrow_F$ $\leftarrow_F$	Residuation $\rightarrow_W$ $\leftarrow_W$	Residuation $\rightarrow_{AM}$ $\leftarrow_{AM}$	Schemas ⓒ	Schemas ⓒ, $\rightarrow_{W,F}$	Schemas ⓒ, $\rightarrow_{AM}$
$\wedge_F$	**PCON**	✓	✓	✓	×	×	×
$\vee_F$		×	×	×	×	×	×
$\wedge_I$	Wansing’s method 1	✓	✓	×	×	×	×
$\vee_I (= \vee_F)$		×	×	×	×	×	×
$\wedge_c$	Wansing’s method 2	✓	✓	×	×	?	✓
$\vee_c$		×	×	×	×	×	×
$\wedge_N (= \wedge_F)$	Francez’s method	✓	✓	✓	×	×	×
$\vee_N$		×	×	×	×	×	×

semantics of which is constrained by a binary relation R on the set of all formulas. The truth condition for relating implication imposes the relatedness constraint as follows:

$A \rightarrow^{W} B$ is true iff A is not true or B is true, and A and B are related by R.

(R has to satisfy certain conditions, which I will not reproduce here, in order to deliver connexivity.) Then, one could consider conjunctions and disjunctions suitable related by some R to, say, validate the parameterized schemas in Section 4.

Wider poly-connexivity Taking the validity of the schemas in the introduction as necessary conditions for connexivity leaves out some logics as merely "demi-connexive", for example Priest's in [18], or Francez's $\mathbf{N}^{\sim\imath}$ in [5], which lack Boethius' Theses in arrow form, even in Pizzi's weak form. It can certainly be debated whether those are necessary conditions, and whether a wider, more open definition of 'connexive logic' is needed. If a wider, although relatively well-defined, notion is adopted, that would re-shape a lot of debates in the field of connexivity, not only this one about poly-connexivity. And I do not take this to be a bad thing at all to be honest.

Other bearers of connexivity Another possibility is that all this talk about connexive connectives is misguided, and that we should stick to connexivity as a property of formulas in general or perhaps of logics instead. This was a path followed by Avron [3] in the case of relevance.

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