Descartes &
The Birth of Formal Objects[1]

Axel Arturo Barceló Aspeitia

Instituto de Investigaciones Filosóficas, UNAM

1. The Problem of Mathematical Objects

a. Foundational Ontology

Twentieth Century (analytic) philosophy of mathematics placed mathematical objects at the center of their foundational program. Ever since Frege's seminal *Foundations of Arithmetic* (1884), the central debate in the field has been between Platonism – the view that mathematical objects exist as abstract objects – and different varieties of Anti-Platonism: psychologism, nominalism, immanent realism, and Meinongianism (Balaguer 1998). Different epistemological, logical and semantical accounts of mathematics have all set themselves forth in relation to their ontological consequences (Shapiro 2000). At least since Benacerraf's deeply influential paper "Mathematical Truth" (1973), for example, the central epistemological problem for mathematics has been to account for our human capacity – as finite material beings – to have reliable knowledge of abstract mathematical entities. A similar semantic problem can be raised about our linguistic capacity to reliably refer to them. It is no exaggeration to say that the putatively abstract nature of mathematical objects has defined the philosophical agenda in the field, and few philosophers have dared to venture far from the central questions about mathematical objects:

- What (and which ones) are they?
- What kind of properties do they have?
- How can we know anything about them?
- What kind of (epistemic) access do we have to them?

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- How do we develop concepts to think about them?
- How can we talk about them? What can we say about them?

b. Abstract Objects

This foundational kind of analytic philosophy of mathematics has taken what I considered a misjudged path to develop an appropriate ontology for mathematics. It is based on two apparently intuitive theses that create a fundamental tension: that mathematical objects and everyday middle-sized physical objects are both somehow similar and deeply different. Both are similar, in so far as they are objects in the very same sense, and those that exist, do in the same single sense of existence (cf. Azounni 2004, Burgess & Rosen 1997, Panza forthcoming). The main difference between them, in contrast, is commonly stated in terms of each kind being respectively paradigmatically abstract and paradigmatically concrete. Today, despite the many difficulties in drawing a sharp boundary between abstract and concrete objects (Burgess & Rosen 1997), "it is universally acknowledged that numbers and the other objects of pure mathematics are abstract, whereas rocks and trees and human beings are concrete." (Rosen 2001) This means that they possess the paradigmatic properties of this ontological category, i.e. they are:

- Not spatio-temporally located
- Causally inert (they are neither causes of anything nor caused by anything)
- Ideal, and
- Formal

The goal of this brief paper is to advance a study on how this image of mathematical objects has historically emerged. Think of it as a Wittgensteinean exercise on looking for or whether 'language went on a pic nic' while we developed the idea that mathematics deals with a specific type of objects[2]: abstract objects. In particular, I will focus on the development of the idea that mathematical objects are formal. I am interested in the sense in which mathematical objects are said to be formal. But, ideally, the answer to this question must throw some further light on how

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this peculiar feature of mathematical objects is related to the other aspects of mathematical objects' putative abstract nature.

Before even trying to answer a question like 'do mathematical objects exist?', it is imperative to determine whether such a question makes any sense. This Carnapian strategy has been recently pursued by Azzzouni (1994), Burgess and Rosen (1997), among others. However, while Azzouni and most other philosophers of mathematics have focused on the notion of existence (or being there) involved in the question, I would like to follow Burgess and Rosen (1997, 11) in analyzing first and foremost the notion of mathematical object, especially the characteristics that such objects, were they to exist, would exhibit. In this article, as I have mentioned, I will focus on their formal character.

2. Descartes & The Birth of Formal Objects

It would be an impossible – and rather pointless – task to try to designate a particular date or event as the birth of formal objects. It makes little sense to draw a line between a time when mathematical objects were worldly, and a later time when they became fully formal. Nevertheless, it is still illuminating to single out certain episodes as particularly meaningful within this long process. Within this history, Descartes' analytic method may be correctly singled out as bearing birth to modern formal mathematical objects. These new objects lived alongside the previous, more traditional objects for a long time during the development of modern mathematics. It was not until the XIX and XXth centuries that formal objects overtook the whole of mathematics. In this paper, I will not cover this later phenomenon3, but the emergence of the first early formal objects. In other words, my aim is not to explain how mathematical objects became formal, but how the first formal mathematical objects were born.

a. Descartes' Ingredients

The main object of my project is to single out the main ingredients that allowed Descartes to develop a new conception of mathematical objects, and explain how they were combined into

[2]. What mathematical true statements are about, what mathematical knowledge is knowledge of.

this new conception. These elements were already present, and many of them developed by other mathematicians and philosophers besides Descartes. However, their conjunction is especially salient in the work of the French mathematician. These are:

1. Analysis as universal method
2. The synthesis of regressive, decompositional and transformative modes of analysis in a single conception
3. The algebraic lingüistic turn
4. The use of abstract variables in terminal terms

In the following sections, I will develop each element and, finally, in the last section will join them together in a simple and straightforward argument that gives us what we want: the birth of formal mathematical objects.

b. Analysis as universal method

Omnia apud me mathematica fiunt
Descartes (Attributed)

It may seem like a controversial thesis to maintain that Descartes' analysis purported to be a universal method. After all, the opening remarks of his Geometry (1637) explicitly expresses as the goal of his analytic language, unlike the previous algebraic language of Vietá,[4] to have a single intended interpretation in geometrical magnitudes. Nevertheless, there is further evidence that, even this remark is better understood in universal terms. In his famous letter to Isaac Beeckman from 1619, he is very explicit about his goal of formulating a universal method "by which all questions may be resolved as regards any sort of quantity, continuous or discrete." (1965, X, 156-158). This must be understood in the context of what Descartes understands as a quantity, which is, in short, everything:

> I openly acknowledge that I can point to no other kind of thing than that which can be divided, shaped, and moved in all kinds of way, and that geometers call quantity and take as the object of their demonstrations." (Principles Pt. II §64, 1965, IX-2, 102)

[3]. A path excellently followed by Corry (2003)
[4]. Cf. Panza (2006)

Descartes' quest for universality, in method and notation, is very closely linked to its unifying impetus.[5] Application establishes the crucial link between unification and universality. For a method to be universal, it is not enough for it to be topic-neutral (that is, it is not enough for it to be non-particular), it must also be universally applicable (it must still be broadly applicable to different particular fields and objects, even if it is strictly not about them, that is, even if it says nothing *particular* about them). Thus, there is a negative and a positive dimension to universality. On its negative aspect, being universal is the opposite of being particular, i.e. not being about anything in particular. From its positive side, universality is the broadest generality. Without being about anything, it is still – in a different sense – about everything. It must be applicable to everything.

The philosophical project of developing a universal method and a universal language for mathematics, both at the times of Descartes and today, exploits both dimensions of universality. The negative dimension of universality, its topic-neutrality, serves to obtain a desired purity of knowledge. Many foundational issues in mathematics have been formulated in these terms: mathematics cannot be on firm grounding unless we remove from its method and its language any element coming from any particular of its fields.

The positive dimension, on the other hand, allows for mathematics to be broadly applied. Even before naturalists of the XXth Century started looking at application as a primary foundation for mathematics, it was understood that only a universally applicable mathematics – that is, a universally applicable mathematical method and notation – could serve as the basic foundation for the whole of mathematics – that is, for all particular mathematical fields. The positive dimension of mathematical universality gives it its unifying power.

The goal of developing a formal foundation for mathematics – including a formal regime of representation – therefore, was to keep mathematics pure but undivided. What Descartes sought in his method of analysis was a system of universal representation in which representatives would stand neither for any particular mathematical sector, nor much the less for any non-mathematical element, but for all of them.

[5]. Thanks to Guillermo Zambrana for stressing this point to me.

c. Descartes' Analytic Synthesis[6]

According to Michael Beaney (2002, 2003), throughout western modern philosophy, the notion of analysis has manifested itself in three different conceptions or modes: regressive, decompositional and transformative. Cartesian analysis, perhaps for the first time synthesizes all the aforementioned modes of analysis: regression, decomposition and transformation.

Descartes inherited the regressive mode of analysis from ancient Greek Geometry (particularly, from Pappus' commentary on Euclides[7]). In this mode – which more or less corresponds to what Hintikka and Remes (1974) call directional analysis –, analyzing a problem "involves working back to the principles, premises, causes, etc., by means of which something can be derived or explained." (Beaney 2002, 55) Beaney calls this mode 'regressive', because of its inverse direction regarding its complementary method of synthesis. However, it is important to identify two different senses in which analysis may be characterized by its 'inverse' direction. On the one sense, analysis goes in the inverse direction with respect to synthesis. Thus, synthesis just traces forward the steps analysis laid out for it. On the second sense, analysis works backwards regarding the direction of logical consequence. Under the otherwise reasonable assumption that what is sought is a deductive proof (such that synthesis ought to follow the direction of logical consequence, from axioms, definitions and postulates to theorems), both senses become equivalent. However, such assumption stops being reasonable, once we leave such cases behind (cases which were not even paradigmatic neither in ancient nor in cartesian geometry). First of all, in constructive cases (where what is sought is the construction of a figure) it makes little sense to talk about a logical direction among concomitants [akóloytha]. Second of all, in theoretical cases (where what is sought is a proof for a theorem), regressive analysis is, above all, a deductive hypothetical method, not an abductive one.[8] Thus, this is the most natural way to read Pappus remark that proof is the reverse of analysis[9], and, later, Alexander of Aphrodisias claim that analysis is the return from the end to the principles.

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6. The following section borrows heavily from my Barceló (2004).

7. The mathematical source of the term 'analysis' has been recognized at least, since (Blancanus 1615), Waitz's comments to the translation of Aristotle's *Organon* (1844-46, I 366) and (Solmsen 1929). Cf. Einarson (1936, 36).

8. For more on the relation between analysis and abduction see Aliseda (2006)

9. On Pappus' *Mathematical Collection*, composed around 300 AD.

(Gilbert 1960, 32 apud. Beaney 2003) The directions of proof and analysis are mutually inverse, not because only one of them follows the natural direction of logical consequence, but only because the starting point of one is the final point of the other. The conclusion of the synthetic proof is the hypothetical premise from which analysis starts. It is only in this sense that analysis is said to "work backwards".

Thus, instead of stressing the putative 'inverse' direction of analysis, it is better to characterize the regressive mode of analysis by its hypothetical and foundational dimension: (i) it starts with an assumption of already having what is sought and (ii) arrives to the principles, premises, causes, etc., by means of which something can be derived or explained. This is what Volker Peckhaus (2002) has called the "foundational" sense of analysis.

Bealey's second mode of analysis, the decompositional, has deeper philosophical roots,[10] since it is a direct descendant of Plato's mature method of collection and division (as it appears in the Phaedrus, Sophist, Politics and Philebus), where concepts are analyzed – decomposed, that is – into other more general concepts.[11] A similar mode is observed in the Aristotelian method of definition through genus and specific difference.[12] So, for example, the concept of human being is decomposed in the concepts of animal and political. Even though the latter are extensionally broader than the original concept, this later concept intensionally contains them, in so far as its definition presupposes them.

However, it is the third mode of analysis that that gives our modern conception its idiosyncratic sense. Bealey calls this mode, 'transformative' because it involves paraphrasing or changing the problem's representation.[13] Even though there is a transformative element easily

[10] Nevertheless, it is reasonable to assume that ancient Greek geometry had a strong influence on both Plato and Aristotle. Cf. Beaney (2003) Therefore, all three modes of analysis have strong mathematical roots. Cf. Benedict (1936, 36-39).

[11]. Furthermore, Beaney (2003) finds in the decomposotional mode another bridge between the formal and the analytical. In his interpretation, Plato's method of dihairesis lays the basic groundwork not only of the decompositional mode of conceptual analysis, but also of its formal dimension. Although Plato did not use the term 'analysis'– his word for 'division' was 'dihairesis' – its goal was finding of the appropriate 'forms' and, subsequently, laying down synthetic definitions.

[12] Aristotle follows a decompositional kind of analysis in his analysis of figures. (*An. Pr.*I32, 42-10) See Benedict (1936,39)

[13] It is very important not to confuse the use of the term 'representation' in contemporary philosophy of science, and in the philosophy of mind and language. In this paper I restrict my use of the term to the first sense.

identifiable in ancient geometrical analysis,[14] it only acquires a special significance in modern analysis. It was Rene Descartes who turned analysis into a formal transformational method.

Taking classical geometric analysis as paradigm,[15] Descartes' analysis aims to find the fundamental principles to build all knowledge –either geometrical or philosophical – upon in a synthetic fashion.[16] Given that, he could find very little explicit information about this method in the available classic texts,[17] his reconstruction is, instead, the creation of a new analytic method.[18] This new method includes as much a change of representation as a method of regression[19] and decomposition.[20] In his *Geometry*, Descartes creates a new formal framework for the representation of geometrical problems. An essential element of this framework is the use of an algebraic language. As we will see in further detail, this change of notation is in itself a radical revolution in mathematics. Still, Cartesian method also involves an element of decomposition. However, such decomposition is completely dependant on the change of notation. Thus, Cartesian analysis synthesizes the three modes of analysis in a single method.

The idea of Cartesian method as a revolutionary change in scientific representation is already found in authors as diverse as Martin Heidegger (1977), Ernst Cassirer (1957), Michel Foucault (1970) and Jonathan Crary (1990), all of whom place it at the very origin of modern thought. Beaney's study, on the other hand, goes one step further by analyzing this modern notion into its decompositional, regressive, and transformative components. In other words, while the previous authors succeeded in identifying a transformative element in Cartesian analysis, they had not separated it from its regressive and decompositional elements.[21] Thus,

[14] Beaney quotes Hankel 1874, 137-50 and Heath (1921) I, 140-2) It is especially clear in Aristotle, whose *Analytics* show very sophisticated syntactic methods of transforming a syllogism's structure.

[15] It is important to remember that Descartes' *Geometry* was originally published alongside the *Discourse of Method* as a sample application of such method.

[16] Descartes points out the simmilarities between his method and classic geometrical analysis in (1965) VII 424, 444-5, (1985, 1984, 1992) I 18-19, II 5, 111. Cf. (Flage 1999, 3) François Viète, the first to introduce variables to geometrical analysis, was of the same opinion.

[17] Descartes accuses the classics geometers of hiding their method of analysis in (1965) X 336, (1985, 1984, 1992) I 19 and (1965) VII 157, (1985, 1984, 1992) II 111.

[18] Even though it maintains a strong continuity with Pappus' method. Compare Pappus' definition with Descartes' in his *Geometry* (1965 VI 372)

[19] In the preface to the French edition of the *Principles* (1965 IXB 5, 1985, 1984, 1992 I181), Descartes describes his method of analysis as the search for 'first causes'. See (Flage 1999, 1, 14)

[20] See (Flage 1999 32-43)

[21] Another important difference between these authors' interpretations, and Beaney's (and mine) is the strong emphasis they place on *order* in Cartesian analysis. True, Descartes stresses the importance of order in passages like (1965) X 379, 451, VI 21, VII 155, (1985, 1984, 1992) I 64, 121, II 110. See (Flage 1999, 38-43). However, a closer reading of these passages shows that order is not important for

they had failed to isolate the actual innovation that defined modern analysis. As Beaney correctly points out, the distinction is essential to understand the truly innovate aspect of modern analysis. Both regression end decomposition had always been essential elements of analysis. It was the transformative element that Descartes deeply transformed – no pun intended.[22]

The following section dives deeper into those waters. In order to make better sense of this revolution, I will differentiate between mere symbolic representation and full-fledged formalization, stressing the crucial role that the introduction of abstract variables played in the development of formal notation.

d. Algebraic Notation[23]

It is easy to notice that the representational regime that operates in Cartesian analysis is not only symbolic but algebraic and abstract. Before Viéta's introduction of abstract variables, mathematical symbolism hardly featured any means of expressing general calculations. Generality was expressed mostly through particular cases that worked both as examples and paradigms. It was not until the work of Viète and Descartes[24] that proper algebraic variables appeared in modern mathematics. Their introduction allowed two important advances in mathematics: the possibility to express general forms – 'species', in Viète's terminology – and,

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analysis, but for (mathematical) induction. Descartes himself recognizes this in (1965) X 388-9, (1985, 1984, 1992) I 25-6.

[22]. That is why this paper focuses so much on formal representational. Despite having already identified some representational change in formal analysis, Beaney does nothing to characterize it or contrast it with similar representational regimes in the history of science.

In contrast, in (2003), Beaney attributes the origin of the hegemony of the decompositional mode in modern philosophical thought to Descartes. In particular, he traces it back to rule thirteen of the Rules for the Direction of the Mind, which states: "in order to perfectly understand a problem we must abstract from every superfluous conceptions, reduce it to its simplest terms and, by means of enumeration, divide it up into the smallest possible parts" (I, 51), and, later, to the second rule for his philosophical method presented in the Discourse on Method, where he instructs "to divide each of the difficulties I examined into as many parts as possible and as may be required in order to resolve them better." (I, 120)

Beaney stresses as an interesting fact that Descartes' Geometry was first published together with the Discourse and advertised as an essay in the method laid out in the Discourse, for each part was responsible for the rise of a different mode of analysis on separate sides of the mathematics/philosophy divide that Descartes was trying to bridge. Thus, in early modern times, the decompositional account would become standard among philosophers, while the transformative mode revolutionized mathematics.

[23] The following section is an abbreviate version of the history of formal analysis in (Barceló 2004)

[24]. With important contributions from Harriot, Girard, Oughtred and Hudde. See Kline (1972) 259-63.

even more importantly, the possibility to calculate with them.[25] In this respect, Kline (1972) has written:

> Viète was completely aware that when he studied the quadratic general equation ax2 + bx + c = 0 (in our notation), he was studying a whole class of expressions. By differentiating numerous from specious logistic in his Isagoge, Viète also differentiated between algebra and arithmetic. Algebra, the specious logistic, he said, was a method of calculation with species or forms. Arithmetic –the numerous– deals with numbers. So, in just one step, algebra was turned into a study of general types of forms and equations, given that what is done for the general case covers an infinite of special cases. (1972, 261-2)[26]

Thus, the central difference between modern and ancient algebra was that, through the use of variables, the former could abstract the common form of particular calculations and express it in a general formula. This new symbolic language allowed mathematicians to manipulate general forms in ways that were nearly impossible until then.

3. An Early Geometrical Example of Formal Analysis

Unlike philosophers, most mathematicians soon recognized the value of Descartes' new method. His famous solution to the 'three or four lines problem'[27] demonstrated its effectiveness in the mind of many modern mathematicians. It would be good, then, to take a closer look at this solution to illustrate the very important role the transformative way of analysis plays in this kind of analysis, so to understand the radical change that formalization represented in the development of mathematical analysis.

The problem is posed the following way.[28] Being AB, AD, EF and GH straight lines given as in the following figure:

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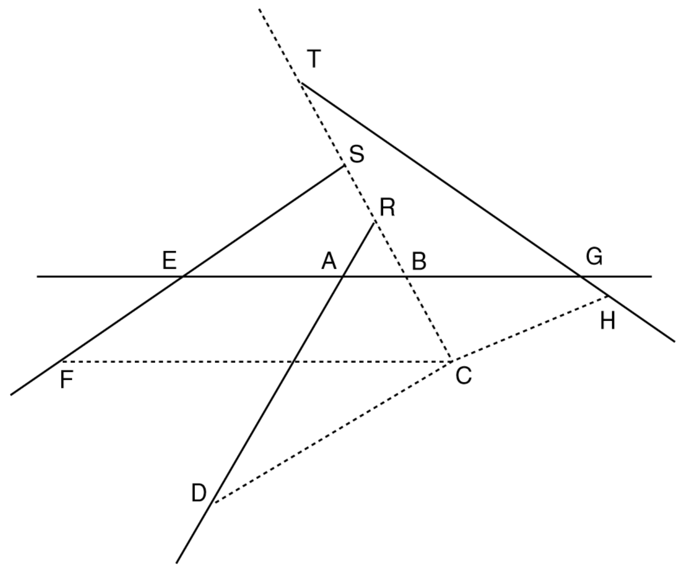
[25] In modern mathematics, talk of 'generality' must not be understood in the same inductive sense it has outside of mathematics. Instead, every formal statement is mathematically 'general', in so far as it is a general schema for expressions or calculations of the same form. Thus, it would be justified to say that, in mathematics, one does not *generalize*, but *formalizes.*

[26]. For Kline, Viète's introduction of variables was "the most significant change in the character of algebra" during the XVI and XVII centuries (Kline 1972, 261)

[27] According to Pappus, this problem had been discussed, but not solved, by Euclides and Apolonius.

[28] I take the reconstruction of the problem from (van der Waerden 1985, 74-5).



Search all points C such that line segments CB, CD, CF and CH, drawn from C to the given four lines, satisfy the following condition: that the product of CB times CD is in a given proportion to the product of CF times CH. It is also asked whether such points are located inside a conic section – a circle, parabola, hyperbola, ellipse or similar –, or not.

In his analysis of the problem above, Descartes starts by assuming that the condition is satisfied, that is, that such a C point exists. Up until here, the method follows closely Pappus definition of regressive analysis, according to which the first step is to assume that which is sought. However, the way Descartes represents this supposition is what differentiates his method from classical analysis. While, in classic analysis, point C is represent by a point in a geometric figure (similar to the one with which I have illustrated this problem), Descartes represents C by a pair of algebraic coordinates. Given that C is determined by the length of segments AB and BC, given angle ABC, it can be modeled by an ordered pair (x,y), where x and y correspond to the aforementioned lengths.

Let me stress again the revolutionary change of representation that Descartes performs here. To represent geometric hypotheses, classical figurative analysis could work with, at most, particular instances what was sought to be constructed or demonstrated in a general way (just like pre-formal algebra). This risked founding some posterior inference in the particularities of such instance, instead of the general specifications of the problem. The introduction of algebraic variables solved such problem. The use of variables allowed Descartes to represent his hypothesis in a formal, algebraic and universal way. In strict sense, the pair of Cartesian coordinates does not represent any particular point, but the general form of a point. In the current example, the introduction of algebraic variables allowed Descartes to represent, in a single view, all C points that constituted the general solution to the problem. This way, his analysis acquired the necessary formal and general character.

The next step is to show that all segments CB, CD, CF and CH are lineal functions of x and y.[29] If so, the original condition of proportionality between CB·CD and CF·CH can be expressed by a quadratic expression with two variables. Each pair of coordinates (x,y) satisfying the equation would represent each of the C points that are sought.

By representing the set of C points in a quadratic equation, one does not only algebraically represent the original geometric concept, but also formalizes it. That way, it becomes possible to know the kind of conic section that such points stand on, attending merely to the equation's syntactic form. In the words of W. Rouse Ball (1908), Descartes discovered that

> It was at once seen that in order to investigate the properties of a curve it was sufficient to select, as a definition, any characteristic geometrical property, and to express it by means of an equation between the (current) co-ordinates of any point on the curve, that is, to translate the definition into the language of analytical geometry.

Notice, therefore, that the quadratic equation does not represent any particular geometrical curve, but a general form of solution to a general problem, and furthermore, that its syntactic features – qua features of an equation, not of a curve as previously understood – become geometrical, that is, mathematical properties. The equation, therefore, becomes a new *formal* kind of mathematical object, radically different from the previously recognized geometrical objects, like curves, points or lines.

[29] Descartes achieves this through the algebraic calculation of the arithmetic relations between *AB*, *BC* and the aforementioned lines. Notice that, since the segments are represented in function of coordinates *x* and *y*, these calculations are neither geometrical, nor arithmetical, but algebraic.

4. The Representational Problem of Analysis

Generalizing from the above example, we can characterize Cartesian analysis as mainly a representational problem. It is my contention that, for Descartes, and for many subsequent mathematicians and philosophers,[30] analysis sought to represent something not given, and to include in such representation whatever available information about the non-given object or property may be relevant for the solution of the problem under analysis, in such a way that the problem may be resolved by mere inspection, transformation and decomposition of such representation. It was thus Descartes' insight that the proper 'analytic' representation of a problem must be one such that its fundamental elements are represented as parts, structured in such a way that the solution becomes evident from mere inspection and transformations on such representation.

The development of algebraic tools for the resolution of mathematical (i.e. geometrical and arithmetical) problems led to a new representational regime in mathematics, that may for the first time be properly called *formal.* Thus, formal mathematics may be best characterized as the solution of mathematical problems and the determination of properties of mathematical objects through the analysis of their representations. This sort of *linguistic turn* in mathematics is what allowed for a new conception of mathematical objects, where representation led to ontology. However, in order to fully characterize this link between ontology and representation in formal mathematics, there is still one further element to consider: the distinction between terminal and problematic terms.

a. Terminal and Problematic Terms

Even though there is no explicit formulation of the distinction between terminal and problematic terms in the current philosophical or historical literature, different version of this distinction can be found in the work of Ludwig Wittgenstein (1974), Saul Kripke (unpublished), and Marco Panza (forthcoming). The basic idea is that every mathematical calculus contains a basic (conventional) distinction between the way problems are represented and the way

[30]. Clearly in the case of early analytic philosophy (Barceló 2004)

solutions are represented. This yields a parallel distinction between terminal and problematic representations. At the beginning of a problem or calculation, the object or property to be found or constructed must be somehow characterized, that is, represented in a way that is both explicit – it must contain every feature that may be relevant to the problem at hand – and, yet, somehow still problematic. Not any way of representing mathematical objects may fulfill this role. We will call the kind of representations that do "problematic". Similarly, in the solution, the object or property sought or constructed must again be fully represented in a way that is still explicit but no longer problematic. This again restricts the way such objects or properties may be represented. Representations of this kind we will call "terminal".

In elementary school arithmetic, for example, decimal notation is usually used for the canonical representation of numbers. This means that for these arithmetical problems or calculations, the final result must be expressed in its proper canonical notation. Numerals in decimal notation, therefore, play the role of terminal terms in the calculus. Algebraic expressions or other numerical expressions, for example, are not acceptable as solutions to elementary arithmetical problems, even if, in strict sense, they may represent the same correct numbers. If a child is asked to supply the addition of thirty five plus five, for example, she would be wrong in giving "35+5" or "30+10" as answers, even if these later expressions do refer to the same correct number, to know, forty. The child must give the answer in the stipulated canonical way, that is, "40". "35+10", "30+10", etc. may be acceptable *middle* terms in the calculation, but they cannot be offered as final representations of the solution. Instead, they are commonly interpreted as leaving some work or calculation to be done. That is why they are still problematic, not terminal. "35+5" and "5" may refer to the same number, but they do not play the same role in the calculus. Expressions like the first may be used to present an arithmetical problem – for they represent the sought mathematical object in an explicit and determinate way, supplying enough information to identify it –, while expressions of the second sort are acceptable as final representations of the same sought object. In other, plain words, it is only when the object is represented in a canonical or terminal way that one may say that the object has actually been found.

To different calculus correspond different terminal and problematic representations. The goal of calculation – the general form of a mathematical problem – is the transformation of problematic representations into canonical ones, according to the rules of the calculus. The goal of a mathematical calculus, accordingly, is to develop a system of representations where this is possible, i.e. where problems are 'easily' represented by problematic expressions, problematic representations are 'easily' transformed into terminal ones, by the application of the rules of the calculus, so that suitable solutions are given in terminal terms.

b. Reference and Ontology

Marco Panza (forthcoming) has advanced the thesis that a semantic difference rides on top of the distinction between terminal and problematic terms. According to him, mathematical objects are referred to *de re* by terminal terms and *de dicto* by problematic ones. The intuition behind Panza's conjecture is that canonical mathematical terms play the role of proper names relative to a calculus, so that problematic terms are better understood as definite descriptions, in such a way that the ontological commitments associated to each sort are different. A calculus is ontologically committed to the existence of the reference of its canonical terms. Therefore, the choice of canonical representations plays a central role in determining a calculus' ontology.

From Panza's thesis, there is but one simple step to realize that representational changes in mathematics result in ontological changes (a claim also made recently by Madeline Muntelsbjorn). In other words, changes in the ways in which mathematical objects are represented result in changes in what mathematical objects may be. It is my contention that the introduction of formal mathematical representation – through the use of abstract algebraic variables – resulted in the birth of formal mathematical objects.

c. Abstract Canonical Representations

Summarizing our previous account of abstract variables, we have highlighted that they allow for a new kind of mathematical expressions which do not refer to particular quantities, magnitudes or their properties, but allow for (i) the representation of generality through formulae, i.e.

general forms of mathematical expressions, and (ii) the manipulation of such general representations (to calculate with forms). Given the universality of analysis sought by Descartes, we may add that the introduction of such abstract variables permitted the development of a universal analytic language, i.e. a language not tied to any particular interpretation, and available for any sort of mathematical analysis.

Once all the ingredients are in place, it is not difficult to trace the emergence of proper formal objects. First of all, if formal expressions are allowed as terminal terms, that is, if abstract variables are allowed to occur in the terminal expressions of a calculus or theory – as Descartes allowed himself to do in the aforementioned geometrical problem –, then such formulae must refer de re to the proper objects of such mathematical theory or calculus. Since the occurrence of abstract variables in a terminal term cannot be accounted for by appeal to particular quantities or magnitudes, the required objects must, therefore, be of a different formal nature than quantities or magnitudes. Thus abstract formal objects are born!

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