Chapter 2

On Mathematical Objects and Concepts

I. Introduction

This chapter expounds two essential notions in Wittgenstein's grammatical picture of mathematics: mathematical objects and concepts. Throughout the *Big Typescript*, *Philosophical Grammar* and *Philosophical Remarks,* Wittgenstein develops his ideas on mathematics largely through examples from elementary arithmetic. Following his own presentation, this chapter explores Wittgenstein's ideas through his analysis of numerical expressions. Wittgenstein's ideas respond to Frege's seminal work on the concept 'number'. Wittgenstein criticizes Frege on three important and interrelated matters: (1) the concept-object distinction, (2) the view of equations as identity statements, and (3) the Context Principle.

1. The distinction between concept and object lies at the root of Frege's theory of numerical statements in *The Foundations of Arithmetic* (1884). In his 1892 article, *Concept and Object*, Frege presents the philosophical distinction between concepts and objects as semantical counterpart to the grammatical distinction between proper names and predicates. Wittgenstein bases his major criticism of Frege on this distinction. For Wittgenstein, the distinction between subject and predicate upon which Frege builds his analysis is too crude. It overlooks important logical distinctions among concepts and objects. According to Frege, every statement of true subject-predicate form says that an object – the subject's referent – falls under a concept – the meaning of the predicate. For Wittgenstein, mathematical propositions, even when expressed in subject-predicate statements, do not describe objects conceptually. Instead, they display grammatical relations. Mathematical terms correspond to grammatical categories.

2. For Frege, as well as Frank Ramsey, mathematical equations are identity statements. According to them, arithmetical equations express the reference identity of two numerical expressions. For example, the arithmetical equation ‘3 + 4 = 7’ means that numerical terms ‘3 + 4’ and ‘7’ refer to the same object, number seven. For Wittgenstein, mathematical terms are not names. Numerical expressions like ‘3 +4’ and ‘7’ are not names of numbers; not of the same number, not of any number. Since they do not refer to anything, saying they have the same referent is nonsense. Furthermore, arithmetical expressions like ‘3 + 4’, ‘the sum of three and four’ and numerals like ‘7’ belong to completely different grammatical categories. Terms of the first sort express calculations, while those of the second sort express their final results. Saying that they are both numerical expressions means only that they both belong to the same grammatical system, one which *operates* with numbers, i. e. arithmetic. For Wittgenstein, in general, mathematical equations like ‘3 + 4 = 7’ and ‘the derivative of $3x^2$ is 6x’ connect calculations with their final results.

According to Wittgenstein, Frege and Ramsey’s account fails, because treating numerical expressions as names separates numerical identity from syntactic identity. Ramsey and Frege confuse the issue of different numerical expressions *referring* to the same number with them *belonging* to the same number as grammatical category. Different numerical expressions belong to the same number if they obey the same arithmetic rules.

3. Frege’s and Wittgenstein’s philosophical methods are not very dissimilar. Frege’s is a precursor of Wittgenstein’s grammatical method. And Wittgenstein obeyed Frege’s Context Principle, “Never to ask for the meaning of a word in isolation, but only in the context of a proposition.”[1] However, he also thought that Frege did not take his own Context Principle with sufficient seriousness. For Wittgenstein, not asking for the meaning

1. G. Frege, *The Foundations of Mathematics* (Evanston: Northwestern University Press, 1996), x.

of a word in isolation is not sufficient. Asking for the meaning of a proposition in isolation, outside its larger context of use, is shortsighted too. In the case of mathematical propositions, their contexts are calculi. Every mathematical proposition belongs to some calculus. In every calculus, propositions come in two different sorts. *Calculation propositions* connect calculations with their results. *Specification propositions* designate elements of calculi. In the case of elementary arithmetic, mathematical equations connect calculations with their final results. For example, an arithmetical equation like ‘3 + 4 = 7’ says that adding three to four results in seven. A mathematical proposition like ‘7 is a number’, in contrast, specifies an element in the calculus. From this perspective, mathematical propositions like ‘4 is a number’ and equations like ‘3 + 4 = 7’ are radically different, since they play essentially different roles in the calculus.

II. Wittgenstein’s Criticism of Frege’s Concept-Object Distinction

A. *Zahlangaben*

Wittgenstein devotes Section XI of the *Philosophical Remarks* to what he terms *Zahlangaben*[2]. A *Zahlangabe* is any expression or diagram equivalent to a statement of the form ‘there are *n* *X*s (that are) *Y*’, where *n* is a number, *X* is a common noun and *Y* a adjective phrase. This category covers both empirical statements like ‘There is a man on this island’ and mathematical ones like ‘There are 6 permutations of 3 elements’. Furthermore, it also includes arithmetical equations. For example, the equation ‘3 + 3 = 6’ is a *Zahlangabe*, because it is equivalent to ‘There are six units in 3 + 3’.

> Mann kann auch sagen, der Satz ‘es gibt 6 Permutationen von 3 Elementen’, verhält sich genau so zum Satz ‘es sind 6 Leute im Zimmer’, wie der Satz 3

2. Anscombe’s translation of *Zahlangabe* as ‘statement of number’ excludes important cases like diagrams or numerical displays. The expression ‘numerical display’ is closer to the German, because it includes linguistic statements as well as diagrams. This dissertation uses the original German expression.

> + 3 = 6, den man auch in der Form ‘es gibt 6 Einheiten in 3 + 3’ aussprechen könnte. [PR 117 p. 129]
>
> One can also say that the proposition ‘there are 6 permutations of 3 elements’ is related to the proposition ‘There are 6 people in this room’ in precisely the same way as is ‘3 + 3 =6’, which you could also express in the form ‘There are 6 units in 3 + 3’. [PR §117 p. 139]

Wittgenstein makes it fairly obvious that not all *Zahlangaben* are arithmetical and, furthermore, they are not all mathematical. Wittgenstein offers a definite way to grammatically distinguish mathematical *Zahlangaben* like ‘The interval *AB* is divided into two (3, 4 etc.) equal parts’, from non-mathematical ones like ‘There are four men in this room’.[3] For Wittgenstein, the distinction between mathematical and non-mathematical *Zahlangaben* is grammatical. In one important context, they are not exchangeable without grammatical loss. The relevant context is the direct object of the German verb *zu berechen* [to calculate]. Let *p* be a mathematical *Zahlangabe* and *q* a non-mathematical one. The sentence ‘I calculate whether *p* (is the case/true)’ is correct, but ‘I calculate whether *q* (is the case/true)’ is nonsense. It makes sense to talk of calculating *p*, but not *q*. For example, it makes sense to say ‘I calculate whether there are 5 prime numbers less than 11’ or ‘I calculate whether 3 + 3 = 6’, but not ‘I calculate whether there are four men in this room’ or ‘I calculate whether there are a dozen bowls in my cupboard’. Another way of characterizing this grammatical difference is to let ‘There are *n X*s such that *Y*’ be a mathematical *Zahlangabe*, and ‘There are *m Z*s such that *W*’ be a non-mathematical one. It makes sense to say ‘I calculate how many *X* are *Y*’, but not ‘I calculate how many *Z* are *W*’. For example, it makes sense to say ‘I calculate how many permutations are of AB’ but not ‘I calculate how many bowls are in my cupboard’.[4]

3. Both examples come from PR Pt. XI §115.

4. It might seem that Wittgenstein is wrong in this point. It makes sense to talk about calculating things like the number of bowls in one’s cupboard or whether there are four men in this room. Imagine a case in

Using this grammatical criterion, Wittgenstein divides *Zahlangaben* into two major groups. The first group includes statements like ‘There are 4 men in this room’, ‘There are two circles in this square’ and ‘I have as many spoons as can be put in 1-1 correspondence with a dozen bowls’. The second group contains statements like ‘There are 6 permutations of 3 elements’, ‘3+3=6’, ‘There are 6 units in 3+3’, ‘there are 4 prime numbers between 10 and 20’ and ‘A quadratic equation has two roots’. Only the first group expresses genuine [*eigentlich*] propositions.*Genuine* propositions are asserted and negated,[5] true or false.[6] Statements in the second group do not describe any possible states of affairs. For Wittgenstein, this means that they are not genuine propositions.[7] Mathematical *Zahlangaben* are pseudo-propositions.

This distinction is a special case of the more general distinction between genuine and pseudo-propositions. According to Stuart G. Shanker’s *Wittgenstein on the Turning-Point in the Philosophy of Mathematics,* Wittgenstein’s claim that mathematical propositions are grammatical must be understood in the context of Wittgenstein’s comparison of mathematical and genuine propositions (what Shanker calls “empirical propositions”). In particular, it must be perceived in the context of Wittgenstein’s comparison between “the meaning of a mathematical proposition and its method of proof to the meaning of an

which one knows that there are six bowls in each one of the cupboards’ three compartments. In this case, it is correct to say that one can calculate how many bowls are in one cupboard, or that one knows *by calculation* that there are eighteen bowls in the cupboard. However, for Wittgenstein, in these cases, one has not actually calculated whether it is the case that there are eighteen bowls in the cupboard, but whether *it makes sense to say* that there are. The fact that there are such number of bowls in the cupboard is external to the calculus and, in consequence, cannot be settled by calculation alone. The truth of ‘There are eighteen bowls in the cupboard’ still depends on the truth of other non-mathematical *Zahlangaben*: that there are six bowls in each compartment and that there are three compartments in the cupboard. The calculation allows for the transition between these genuine *Zahlangaben*. The role of calculation in these cases is discussed in depth in Chapter 4.

5. PR §163

6. “We could say: a proposition is that to which the truth functions may be applied.” PR §85

7. PR §117

empirical proposition and its method of verification.” Wittgenstein purports to state the similarities and differences between these two sorts of propositions, their meanings and verification methods. He aims at answering two questions: ‘Why are they both propositions?’ and ‘Why are mathematical propositions not empirical?’ Wittgenstein claim that mathematical propositions are grammatical answers both questions.

B. Frege’s Grammatical Enquiry into the Concept of Number

> Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician.
>
> Gottlob Frege

Wittgenstein starts his *Zahlangaben* analysis from Frege’s own investigation on *The Foundations of Arithmetic* (1884). In this seminal work, Frege defined the notion ‘cardinal number’ through the primitive notion of a concept’s extension or ‘value-range’. The insight behind Frege’s definition is that a cardinal number statement such as ‘There are n φ-things’ predicates the number n as a higher-order concept of φ. Namely, it says that n things fall under φ. Frege defines the cardinal number of concept φ (i.e., the number of φs) as the concept ‘being a concept equinumerous to φ’s extension’. This definition identifies the number of planets as the extension of the concept being a concept ‘equinumerous to the concept of being a planet’. The number of planets is an extension containing all and only those concepts which nine objects exemplify, like the concept ‘being a planet’.[8] Frege writes,

> §46. Um Licht in die Sache zu bringen, wird es gut sein, die Zahl im Zusammenhange eines Urteihls zu betrachten, wo ihre ursprüngliche Anwendungsweise hervortritt. Wenn ich in Ansehung derselben Wahrheit sagen kann: “dies ist eine Baumgruppe” und “dies sind fünf Bäume” oder “hier sind vier Compagnien” und “hier sind 500 Mann,” so ändert sich dabei weder

8. Edward N. Zalta, “Gottlob Frege” *Stanford Encyclopaedia of Philosophy* (Palo Alto: Stanford, 1996) [http://plato.stanford.edu/entries/frege/]

> das Einzelne noch das Ganze, das Aggregat, sondern meine Benennung. Das ist aber nur das Zeichen der Ersetzung eines Begriffes durch einen andern. Damit wird uns als Antwort auf die erste frage des vorigen Paragraphen [von wem durch eine Zahlangabe etwas ausgesagt werde] nahe gelegt, daß die Zahlangabe eine Aussage von einem Begriffe enthalte. Am deutlichsten ist dies veilleicht bei der Zahl 0. Wenn ich sage: "die Venus hat 0 Monde", dem etwas ausgesagt werden könnte; aber dem Begriffe "Venusmond" wird dadurch eine Eigenschaft beigelegt, nämlich die, nicht unter sich zu befassen. Wenn ich sage: "der Wagen des Kaisers wird von vier Pferden gezogen," so lege ich die Zahl vier dem Begriffe "Pferd, das den Wzgen des Kaisers zieht," bei. [p. 59]

> §46. It may throw some light on the matter to consider number in the context of a judgement which brings out the way in which it is in origin applied. While looking at one and the same external phenomenon, I can say with equal truth both "It is a copse" and "It is five trees" or both "Here are four companies" and "Here are 500 men." Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but only my terminology. But that is of itself only a sign that one concept has been substituted for another. This suggests as the answer to the first of the questions left open in our last paragraph [when we make a statement of number, what is that of which we assert something?], that a statement of number contains an assertion about a concept. This is perhaps clearer with the number 0. If I say "Venus has 0 moons", there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the *concept* "Moon of Venus", namely that of including nothing under it. If I say "The King's carriage is drawn by four horses," then I assign the number four to the concept "horse that draws the King's carriage." [p. 59e]

In 1892, Frege expanded his treatment of the 'concept' notion in an article for the *Vierteljahrsschrift für Wissenschaftliche Philosophie*. He used this paper to counteract certain criticisms of the *Foundations*, especially those from Benno Kerry. Frege found that these objections stemmed from a misunderstanding of his 'concept' notion. Previously, in the article "Function and Concept" from 1891, he had defined concepts as functions mapping objects to truth values. Later, Frege needed to clarify his distinction between concepts and objects. In his 1892 presentation, he differentiated them through the grammatical distinction between proper names and predicates. For Frege, names refer to objects, while predicates refer to concepts.

> The concept (as I understand the word) is predicative [*footnote*: It is, in fact, the referent of a grammatical predicate]. On the other hand, a name of an object, a proper name, is quite incapable of being used as a grammatical predicate.[9]

C. The Grammatical Method

Wittgenstein and Frege shared a strong belief in grammar's philosophical significance. Both found that logical distinctions are ultimately grammatical. For Wittgenstein, inventing distinctions not existing in natural language grammar is idle philosophical speculation. For every philosophical category *L*, and every expression *x*, the replacement of 'a' in at least one grammatically acceptable statement by *x* makes sense if and only if (the meaning of) *x* belongs to *L*. In this sense, every significant philosophical distinction and category corresponds to a grammatical one. The grammatical analysis that Wittgenstein endorses in the middle period already appears in many of Frege's arguments. For Frege, grammatical categories are philosophically prior to ontological ones.[10] Two objects or concepts are ontologically different if and only if their names have substantially different grammar. For example, in §29 of the *Foundations of Arithmetic*, Frege argues that the number word 'one' does not stand for a property of objects, because it is not a grammatical predicate.

> Wenn "ein Mensch" ähnliche wie "weiser Mensch" aufzufassen wäre, so sollte man denken, daß "Ein" auch als Prädikat gebraucht werden könnte, sodaß man wie "Solon war weise" auch sagen könnte "Solon war Ein" oder "Solon war Einer." . . Noch deutlicher zeigt sich dies beim Plural. Während man "Solon war weise" zussamenziehen kann in "Solon and Thales waren weise," kann man nicht sagen "Solon und Thales waren Ein." Hiervon wäre die Unmöglichkeit nicht einzusehen, wenn "Ein" sowie "weise" eine Eigenschaft sowohl des Solon als auch des Thales wäre. [Pp. 40, 41]

9. G. Frege, "On Concept and Object" in *Translations from the Philosophical Writings of Gottlob Frege*, Geach and Black eds. (Oxford: Basil Blackwell, 1952), 43.

10. "*Frege [da] prioridad a las categorías sintácticas sobre las ontológicas.*" [Frege gives priority to syntactic categories over ontological ones] Matthias Schirn, "*Los Números como Objetos y el Análisis de los Enunciados Numéricos*," *Análisis Filosófico* XIV, no. 1 (may 1994): 21-40.

> If it were correct to take “one man” in the same way as “wise man”, we should expect to be able to use “one” also as a grammatical predicate, and to be able to say “Solon was one” just as much as “Solon was wise” . . . This is even clearer if we take the plural. Whereas we can combine “Solon was wise” and “Thales was wise” into “Solon and Thales were wise”, we cannot say “Solon and Thales were one.” But it is hard to see why this should be impossible, if “one” was a property both of Solon and of Thales in the same way that “wise” is. [Pp. 40e, 41e]

In this example, the logical category ‘predicate of objects’ corresponds to the context λx (‘Solon was *x*’). Only expressions the word ‘wise’ substitute for in the statement ‘Solon was wise’ stand for object predicates. ‘One’ is not among them. In consequence, ‘one’ does not stand for a property of objects. In a similar fashion, in §38, Frege argues that one is a unique object, because ‘one’ functions as a proper name. It makes sense to say ‘the number one’, but not ‘a number one’. Furthermore, it does not have a plural form any more than ‘Frederick the Great’, ‘the chemical element gold’ or any other proper name. ‘One’ is a proper noun. In consequence, one is a unique object. For Frege, the category ‘unique objects’ corresponds to the grammatical category ‘proper names’.

D. Wittgenstein’s Criticism of Frege’s Grammatical Analysis of *Zahlangaben*

Not surprisingly, Wittgenstein criticizes the grammatical distinction behind the concept and object characterization in Frege’s analysis of *Zahlangaben.* The core of this criticism appears in the aptly titled Appendix 2 “Concept and Object. Property and Substrate” of the first part of the *Philosophical Grammar*. Wittgenstein questions the philosophical value of Frege’s grammatical analysis of statements in the subject-predicate form.

> Begriff und Gegenstand, das ist bei Russell und Frege eigentlich Eigenschaft und Ding; und zwar denke ich hier an einen raumlichen Körper und seine Farbe. Man kann auch sagen: Begriff und Gegenstand, das ist Prädikat und Subjekt. Und das Subjekt-Prädikat Form ist eine Ausdrucksform menschlicher Sprachen. Es ist die Form “ x ist y” (“x y”): “mein Bruder ist groß”, “das Gewitter ist nahe”, “dieser Kreis ist rot”, “August

> ist stark”, “2 ist eine Zahl”, “dieses Ding ist ein Stück Kohle.” [PG Pt. I Appendix 2. p. 397]
>
> When Frege and Russell speak of concept and object they really mean property and thing; and here I’m thinking in particular of a spatial body and its colour. Or one can say: concept and object are the same as predicate and subject. The subject-predicate form is one of the forms of expression that occur in human languages. It is the form “x is y” (“x y”): “My brother is tall”, “The storm is nearby”, “This circle is red”, “Augustus is strong”, “2 is a number”, “This thing is a piece of coal.” [PG Pt. I Appendix 2. p. 202]

This interpretation of Frege appears twice in the *Philosophical Remarks.*

> Eine Schwierigkeit der Fregeschen Theorie ist die Allgemeinheit der Worte ‘Begriff’ und ‘Gegenstand’. Denn da man Tische und Töne und Schwingungen und Gedanken zählen kann, so ist es schwer, sie alle unter einen Hut zu bringen.
> Begriff und Gegenstand, das ist aber Prädikat und Subjekt. Und wir haben gerade gesagt, daß Subjekt-Prädikat nicht *eine* logische Form ist. [PR §93 p. 109]
>
> One difficulty in the Fregean theory is the generality of the words ‘concept’ and object’. For even if you can count tables and tones and vibrations and thoughts, it is difficult to bring them all under one roof.
> Concept and object: but that is subject and predicate. And we have just said that there is not just one logical form which is *the*subject-predicate form. [PR §93 p. 119]
>
> Man kann natürlich die Subjekt-Prädikat – oder was dasselbe ist – die Argument-Funktion-Form als eine Norm der Darstellung auffassen, und dann ist es allerdings wichtig und charakteristisch, daß sich in jedem Fall, wenn wir Zahlen anwenden, die Zahl als Eigenschaft eines Prädikats darstellen läßt. Nur müssen wir uns darüber im klaren sein, daß wir es nun nicht mit Gegenständen und Begriffen zu tun haben als den Ergebnissen einer Zerlegung, sondern mit Normen, in die wir den Satz gepreßt haben. Und es hat freilich eine Bedeutung, daß er sich auf diese Norm hat bringen lassen. Aber das In-eine-Norm-Pressen ist das Gegenteil einer Analyze. Wie man, um den natürlichen Wuchs des Apfelbaums zu studieren, nicht den Spalierbaum anschaut, außer, um zu sehen, wie sich *dieser* Baum unter *diesem* Zwang verhält.[PR §115 pp. 125, 127].
>
> You can of course treat the subject-predicate form (or, what comes to the same thing, the argument-function form) as a norm of representation, and then it is admittedly important and characteristic that whenever we use numbers, the number may be represented as the property of a predicate. Only we must be clear about the fact that now we are not dealing with objects and concepts as the results of an analysis, but with moulds into which we have squeezed the proposition. And of course it’s significant that

> it can be fitted into this mould. But squeezing something into a mould is the opposite of analysis. If one wants to study the natural growth of an apple tree, one doesn't look at an espalier tree – except to see how *this* tree reacts to *this* pressure. [PR §115. Pp. 136, 137]

Wittgenstein complains that the distinction between subject and predicate at the base of Frege's analysis is undeveloped. "The subject-predicate form serves as a projection of countless different logical forms."[11] Frege's analysis is not mistaken, it is limited in scope. It ignores important differences within the 'object' and 'concept' categories. In particular, it fails to distinguish between genuine objects and mathematical ones. It fails to recognize that mathematical concepts are not genuine concepts.

> This notation is built up after the analogy of subject-predicate propositions in ordinary language, such as those describing physical objects. . . And propositions having different grammars, both mathematical and nonmathematical propositions, are dealt with in the same way, e. g., "All men are mortal," "All men in this room have hats," "All rational numbers are comparable in respect to magnitude." [WL *Philosophy for Mathematicians* 1932-33 §1 p.205]

For Wittgenstein, mathematical and non-mathematical *Zahlangaben* involve different concepts. Both sorts of *Zahlangaben* can take the form 'There are n Xs (such that) Y'. However, the concepts in X and Y are different in each. Concepts such as 'persons', 'spoons', 'this room', 'this square', 'my mother's cupboard' usually occur in non-mathematical *Zahlangaben.* Mathematical *Zahlangaben*, on the contrary, contain terms like 'pure colors', 'units', 'permutations' and numerical expressions like '3 + 3', 'two' and 'as many as can be out in 1-1 correspondence with a dozen bowls'. For a *Zahlangabe* to be mathematical, Y must be a calculation concept in the arithmetic of X.[12]

11. PG Pt. I Appendix 2. p. 205.

12. The following section develops the 'calculation concept' notion, while Chapter 4 explains the expression 'the arithmetic of. . '. The present chapter analyzes only elementary arithmetic, where X is the concept 'unit'. This section deals exclusively with pure arithmetic, instead of applied arithmetic.

The terms occuring in non-mathematical *Zahlangaben* are genuine names of objects or concepts. The terms in mathematical *Zahlangaben*, on the contrary, do not refer to any genuine concepts. The objects that fall under them are *improper* [*uneigentlich*].

> Ja, wir sprechen vom Kreis, seinem Durchmesses, etc., etc. Wie von einem Begriff, dessen Wigenschaften wir beschreiben, gleichguultig, welche Gegenstände unter diesen Begriff fallen. – Dabei ist aber 'Kreis' gar kein Prädikat im ursprüchlichen Sinn. [PG Pt. I. Appendix 2. p. 404]

> We do indeed talk about a circle, its diameter, etc. etc. As if we were describing a concept in complete abstraction from the objects falling under it. – But in that case 'circle' is not a predicate in the original sense. [PG Pt. I. Appendix 2. p. 207]

Wittgenstein's distinction between 'genuine' and 'improper' concepts and objects springs from his criticism of Frege's account of *Zahlangaben.* The heart of Wittgenstein's objection is that Frege's distinction between object and concept is an insufficient analysis of mathematical *Zahlangaben.* It misses important conceptual differences between mathematical pseudo-propositions and genuine propositions. Frege was blind to the fact that genuine and mathematical concepts' cardinalities are radically different. Wittgenstein's criticism clarifies this difference.

E. Mathematical Objects and Concepts

First, Wittgenstein objects to Frege's analysis of descriptions. Wittgenstein calls 'description' any statement of the subject-predicate form. According to Frege, every description says that an object, the referent of the subject, falls under a concept, the referent of the predicate. Wittgenstein considers two different kinds of description: *internal* and *external*.[13] Internal descriptions ascribe to objects the properties essential for their existence, while external descriptions ascribe accidental properties to them. For Wittgenstein, a

13. External descriptions answer the question "which . . . ?". Internal descriptions answer the question "what sort of . . . ?". PG Pt. I Appendix 2. p. 398 [P. 204].

property is essential for the existence of an object if its absence “would reduce the existence of the object itself to nothing.”[14] Frege’s analysis holds for external descriptions, but it fails when applied to internal descriptions. In the aforementioned Appendix, Wittgenstein offers the following example:

> Was braucht es zu einer Beschreibung, daß – sagen wir – ein Buch an einer bestimmten Stelle ist? Die interne Beschreibung des Buches, d.i. des Begriffes und die Beschreibung seiner Lage, und die wäre durch Angabe der Koordinaten dreier Punkte möglich. Der Satz “Ein solches Buch ist *hier*” würde dannn heißen, es hat *diese* drei Bestimmungskoordinaten. Die Angabe des Hier darf eben nicht präjudizieren, *was* hier ist. [PG Pt. I, Appendix 2 p. 402]
>
> What is necessary to a description that – say – a book is in certain position? The internal description of the book, i.e. of the concept, and the description of its place which it would be possible to give by giving the co-ordinates of three points. The proposition “Such book is *here*” would mean that it had *these* three coordinates. For the specification of the “here” must not prejudge *what* is here. [PG Pt. I, Appendix 2 pp. 206, 207]

Consider the case in which, pointing at the same object, one makes the following two statements: ‘This book has pages’ and ‘This book is here’. Since having pages is an essential property of any book, the first statement is an internal description. In contrast, the second statement is an example of an external description. It says something about the book. It gives its spatial location. This property is independent of being a book. In the other case, on the contrary, the property of ‘being a book’ already includes ‘having pages’. The internal description does not actually say anything about the object, but about the concept of book under which it falls. The internal description does not describe[15] the book as an object, but the concept ‘book’.

14. PR §94

15. “Property terms in ordinary contexts must stand for qualities that it is sensible to say the substrate *has* or *hasn’t*. It is nonsense to attribute a property to a thing if the thing has been *defined* to have it.” WL *Philosophy for Mathematicians* 1932-33 §3 p. 208.

In this sense, the distinction between internal and external descriptions ultimately depends on the concept under which the described object falls.[16] Instead of an object and a concept, every description involves two concepts. The internal or external nature of the description depends on the relation between these two concepts. If the concept in the subject includes or implies the concept in the predicate, the description is internal. Otherwise, the description is external. The spatial location of a book externally describes it, because the concept of book does not include its location. ‘Having pages’ describes it internally, because the concept ‘book’ includes ‘having pages’. This later case describes the concept ‘book’, not any particular book. In consequence, Frege’s analysis mistakenly says that every description describes some object. For Wittgenstein, internal descriptions do not describe objects. They state conceptual relations.

As in Wittgenstein’s example of the book, the same object under the same concept can be subject of internal and external description. For Wittgenstein, this means that books are genuine objects. Describing genuine objects both externally and internally is possible. However, other objects may only have internal descriptions. Under some concepts no proper objects may fall. These concepts occur only in internal descriptions. In the aforementioned Appendix, Wittgenstein gives shapes and colors as examples of these concepts. In section XI of the *Philosophical Remarks*, he adds mathematical concepts to this list.

> Freilich könnte man so schreiben: Es gibt 3 Kreise, die die Eigenschaft haben rot zu sein. Aber hier tritt der Unterschied zu Tage zwischen den uneigentlichen Gegenständen – Farbflecken im Gesichstfeld, Tönen, etc. etc. – und den Elementen der Erkenntnis, den eigentlichen Gegenständen. [PR §115 p. 126]
>
> You might of course write it like this: there are 3 circles with the property of being red. But at this point the difference emerges between improper objects

16. However, only objects under concepts are describable. Even ostensive definitions work only under concepts. PG Pt. I Appendix 2, p. 340 [p.206].

> – colour patches in a visual field, etc. etc. – and the elements of knowledge, the genuine objects. [PR §115 p. 136]

For Wittgenstein, mathematical objects are not genuine objects. They have no external properties.[17] All their properties are essential, and essential properties do not describe objects. They describe concepts.[18]

All mathematical terms correspond to mathematical concepts. Even in statements of the form subject-predicate, the subject does not refer to any object. For example, the statement ‘4 is a number’ does not ascribe the property of being a number to the object 4. The mathematical statement does not express a proposition of the form $\phi(a)$, because the terms a and ϕ are inseparable.[19] For any mathematical concept, it makes sense to ask whether or not something satisfies it. However, it does not make sense to ask whether or not something that satisfies it *exists*. It makes sense to ask whether or not there are prime numbers between any two given natural numbers, or filters for a determined algebraic structure. However, ‘there are circles’ does not mean that circles exist outside their mathematical role. ‘There are numbers’, ‘circles’, or ‘sets’ only means that those concepts are not extensionally empty. However, the extension of a mathematical concept does not have an existence external to the concept and its intension. In mathematics, ‘there is’ does not equal ‘exists’. Mathematics has no proper *existential* propositions. For Wittgenstein, mathematical propositions of the form ‘$(\ x) \cdot \phi x$’ do not mean that an object x (such that ϕx) exists. Existential propositions only make sense with genuine concepts. Books exist, but numbers do not. Saying that there are numbers only means that the concept ‘number’ is not empty.

[17]. Wittgenstein does not seem to take in account those rare cases where mathematical objects have external descriptions, as in ‘3 is Leroy’s favorite number’.

[18]. PR §94

[19]. A mathematical proposition $\phi(a)$ is true not because a is ϕ, but because it does not make sense to talk about a not being ϕ. That is why Wittgenstein says that mathematical propositions draw the limits of sense. PR §98.

In summary, Wittgenstein thought that Frege's main flaw was not considering his own Context Principle with sufficient seriousness. If he had, he would have noticed that, just like words, propositions have sense only inside a larger context. He would have noticed that different *Zahlangaben* play different roles in language. Some of them are external descriptions – genuine propositions, while others are internal ones – grammatical rules. Mathematical *Zahlangaben* are the latter kind. The objects and concepts in mathematical propositions are completely different from those in genuine propositions. Mathematical concepts are actually grammatical categories. Mathematical objects are not genuine objects, but roles within a calculus.

III. On Mathematical Equations

A. Frege and Ramsey: Mathematical Equations as Identity Statements

> 'Every symbol is what it is and not another symbol'.
> PR §163 p. 196

Frege did not distinguish between mathematical and non-mathematical *Zahlangaben.* All *Zahlangaben* are arithmetical equations. On §57 of *The Foundations of Arithmetic*, he writes:

> Z.B. kann man den Satz "Jupiter hat vier Monde" umsetzen in "die Zahl der Jupitersmonde ist vier.". Hier darf das "ist" nocht als blosse Copula betrachtet werden, wie in dem Satze "der Himmel ist blau". Das zeigt sich darin, daß man sagen kann: "die Zahl der Jupitersmonde ist vier" oder "ist die Zahl 4". Hier hat "ist" denn Sinn von "ist gleich," "ist dasselbe wie". Wir haben also eine Gleichung, die behauptet, daß der Ausdruck "die Zahl der Jupitersmonde" denselben Gegenstand bezeichne wie das Wort "vier". Und die Form der Gleichung ist die Herrschende in der Arithmetik. [P. 69]

> For example, the proposition "Jupiter has four moons" can be converted into "the number of Jupiter's moons is four". Here the word "is" should not be taken as a mere copula, as in the proposition "the sky is blue". This is shown by the fact that we can say: "the number of Jupiter's moons is the number four, or 4". Here "is" has the sense of "is identical with" or "is the same as". So what we have is an identity, stating that the expression

> "the number of Jupiter's moons" signifies the same object as the word "four". And identities are, of all forms of propositions, the most typical of arithmetic. [P. 69e]

Frege viewed numerical equations – and, in consequence, all *Zahlangaben* – as identity statements. In 'On Sense and Reference' [*Über Sinn und Bedeutung*] and *Conceptual Notation* [*Begriffsschrift*], Frege interpreted statements of the form '*a* = *b*' as identity statements. At the very beginning of 'On Sense and Reference', he writes in a footnote,

> I use this word [Equality] in the identity sense and I understand '*a* = *b*' in the sense of '*a* is the same as *b*' or '*a* and *b* agree'.[20]

Under this point of view, arithmetical equations are similar to identity statements like 'the morning star is the evening star'. Just like Frege, Ramsey thought of mathematical equations as identity statements. Unlike Frege, Ramsey echoed Wittgenstein's view of mathematical equality from the *Tractatus* as a relationship between names or signs referring to objects. For Ramsey, they express the referential identity of two nominal expressions. The arithmetical equation '3 + 4 = 7', for example, expresses the referential identity between the numerical terms '3 + 4' and '7'. In other words, seven is the sum of three and four means that the expressions 'the sum of three and four' and 'seven' refer to the same number: seven. In his seminal article *The Foundations of Mathematics*, Ramsey writes:

> . . . in '*a* = *b*' either '*a*', '*b*' are names of the same thing, in which case the proposition says nothing, or of different things, in which case it is absurd. In neither case is it the assertion of a fact; it only appears to be a real assertion by confusion with the case when '*a*' or '*b*' is not a name but a description. When '*a*', '*b*' are both names, the only significance which can be placed on '*a* = *b*' is that it indicates that we use '*a*', '*b*' as names of the same thing or, more generally, as equivalent symbols.[21]

Wittgenstein finds accounts like those of Frege or Ramsey problematic, because by treating numerical expressions as names, they separate the sameness of the number from the

[20]. G. Frege, "On Sense and Reference" (P. Geach and M. Black 1952), 25-50.

[21]. F. Ramsey, *The Foundations of Mathematics and Other Logical Essays* (New York: Harcourt, Brace & Co., 1931), 180.

sameness of the sign. Wittgenstein, in contrast, does not differentiate between numerical identity and syntactic identity. For him, asking when different signs – different tokens of the same sign-type[22]– represent the same number makes no sense. Representing the same number *is* being the same sign.

The question of numerical identity is a question for the identity of sign types. Numerical identity is a grammatical matter.

> Ich meine: Die Zahlen sind das, was ich in meiner Sprache durch die Zahlenschemata darstelle.
> D.h. ich nehme (sozusagen) als das mir Bekannte die Zahlenschemata der Sprache und sage: Die Zahlen sind das, was diese darstellen [(Spätere Randbemerkung): Statt um eine Definition der Zahl, handelt es sich nur um die Grammatik der Zahlwörter].
> Das entspricht dem, was ich seinerzeit meinte, als ich sagte: Die Zahlen treten mit dem Kalkül in die Logik ein. [PR §107]
>
> I mean: numbers are what I represent in my language by number schemata.
> That is to say, I take (so to speak) the number schemata of the language as what I know, and say that numbers are what these represent [(Later marginal note): Instead of a question of the definition of number, it's only a question of the grammar of numerals].
> This is what I once meant when I said, it is with the calculus [system of calculation] that numbers enter into logic. [PR §107 p. 129]

Accordingly, an answer in terms of mere perception is unsuitable. Seeing directly the number a symbol represents is impossible. Just as it is impossible to *see* that the signs 'a' and 'a' are the same letter, to *see* that '|||||||' and '7' are the same number is impossible too.

> Wie kann ich wissen, daß |||||||| und |||||||| *dasselbe* Zeichen sind? Es genügt doch nicht, daß sie *ähnlich* ausschauen. Denn es ist nicht die ungefähre Gleichheit der Gestalt, was die Identität der Zeichen ausmachen darf, sondern gerade eben die Zahlengleichheit. [PR §103. Cf. PG Pt. II §18]

22. Wittgenstein called them 'number schemata'.

> How am I to know that ||||||| and ||||||| are the *same* sign? It isn't enough that they look *alike*. For having approximate similarity in *Gestalt* can't be what is to constitute the identity of the signs, but just being the same in number. [PR §103 p. 125. Cf. PG Pt. II §18 p. 331]

Wittgenstein found that different numerical signs are the same number if and only if they obey the same rules.[23] Determining the numerical identity of different numerical signs requires a grammatical investigation into the rules of the calculus. It must be the result of a 'comparison of the structures' [*Vergleiching der Strukturen* PR §104 p. 117 (p. 126)]. Expressing numerical identity in a mathematical proposition is impossible, because it is not the result of calculation. The problem of numerical identity involves the totality of the calculus. It is not the result of a calculation, but a condition for it. The knowledge that '|||||||' and '7' are the same number does not result from arithmetical calculation. A criterion for numerical identity is necessary for doing arithmetic. However, '3 + 4' = 7' is the result of an arithmetical calculation and does not express the referential identity of the signs '3 + 4' and '7'. Despite both being numerical expressions, they belong to different grammatical categories.

Wittgenstein emphatically rejects the view of mathematical equations as identity statements. Frege bases his interpretation on the view that arithmetical expressions like '3 + 4', or 'the sum of three and four' and '7' are names of numbers and that numerals and other, complex numerical expressions are not philosophically different. For Wittgenstein, mathematical terms are not names. Since they do not refer, it does not make sense to talk about them having the same referent.

Second, arithmetical expressions like '3+4', or 'the sum of three and four' and numerals like '7' belong to different grammatical categories. According to Wittgenstein's method, grammatical distinctions are prior to logical ones. In this case, calculation expres-

23. PG Pt. III §15 p. 602 [p. 307]

sions like ‘the product of 3 by 4’ and result expressions like ‘7’ are not exchangeable in the context of the verb to calculate. In other words, it makes sense to say ‘I calculate the product of 3 by 4’, but not ‘I calculate 7’. Mathematical expressions which can substitute for ‘the product of 3 by 4’ in the aforementioned context stand for calculations. They are both numerical expressions, only because they belong to the same grammatical system *operating* with numbers, i. e., arithmetic.

In German grammar, the verb ‘*berechnen*’ [to calculate] is an active or transitive verb.[24] It requires a direct object. This means that using ‘*berechen*’ without a direct object is grammatically incorrect. Determining the calculation object is essential. In a well constructed sentence of German, the direct object usually follows the verb ‘*berechen*’ in a sentence. Not any sort of expression can serve this function. Answering the question ‘*What* is to be calculated?’ with a verb, adjective or adverb is grammatically incorrect. The answer is always an expression like ‘the successor of four’, ‘the supremum of set E’ or ‘the product of three by four’. These expressions behave like complex nouns. They are either singular or plural. It makes sense to talk about calculating the smallest element in a well ordered set (singular), or about calculating the square roots of a positive number among the reals (plural). And they play the sort of grammatical roles in sentences nouns usually play. Besides the role of direct object, they also play the role of *subject* in certain sentences. For example, it makes sense to say that the supremum of E is less than or equal to any upper bound of E. However, not every mathematical term can play this role. Not any sentence with a numerical expression following the verb ‘to calculate’ makes sense. It makes sense to talk about calculating the product of three by four, but not calculating twelve, for example. It makes sense to talk about calculating the square roots of four, but not about calculating two

24. Unlike the intransitive German verb *rechen*, which also translates to English as ‘to calculate’.

and minus two, even though ‘twelve’, ‘two’, ‘minus two’, just like ‘the square roots of four’ and ‘the product of three by four’ all behave like nouns. In this particular grammatical case, apparently synonymous terms like ‘twelve’ and ‘the product of three by four’ cannot substitute for each other without loss of grammatical correctness.

Mathematical expressions referring to the final results of calculations cannot be direct objects of the verb ‘to calculate’. This latter sort of expressions – sometimes called canonical numerals – behave just like proper names (or lists of them). Talking about *the* number two, instead of *a* number two or *some* number two is grammatically correct. These kinds of mathematical terms are *irreducible.* After arriving at one of them, taking the calculation further is impossible. Instead, the sort of expressions playing the role of direct object of ‘to calculate’ behave more like adjective phrases in natural language. In general, most mathematical terms are either *calculation terms* or *final result terms*. Most philosophers of mathematics conceiving both sort of expressions as names referring to abstract objects overlook this distinction.[25] However, it is essential for the analysis of numerical statements.

B. Equations and Calculations

The distinction between calculation expressions and final result expressions sheds some new light on the distinction between obviously tautological equations like ‘7 = 7’ or ‘3 + 4 = 3 + 4’ and genuine equations like ‘3 + 4 = 7’, which have puzzled so many philosophers like Frege. Despite the superficial similarity, in the second sort of equations, the terms on both sides of the ‘=’ are grammatically different. Call them ‘calculation’ and ‘final result terms’ respectively. One expresses a calculation, and the other its final result. In

[25]. However, not all philosophers have overlooked the distinction. Martin-Löf and Saul Kripke, for example, make a big point of distinguishing terminal and non-terminal mathematical terms.

consequence, the sign '=' in these equations does not work as a *copula.* It connects the calculation with its final result. '3 + 4 = 7', for example, is a mathematical proposition saying that seven is the final result of adding three plus four. In it '3 + 4' expresses the calculation of adding three plus four, while '7' is the final result of such calculation.

By contrast, '7 = 7' is not a genuine mathematical proposition, because it says nothing about any calculation. Seven is not a calculation. Seven is not the result of calculating seven. Such nonsense results from treating identity statements like '7 = 7' as equations on a par with '3 + 4 = 7'.

> Wenn man fragt: Was heißt denn dann aber '5 + 7 = 12' – was für ein Sinn oder Zweck bleibt dann für diesen Ausdruck – so ist die Antwort: Diese Gleichung ist eine Zeichenregeln, die angibt, welches Zeichen entsteht, wenn man eine bestimmte Operation (die Addition) auf zwei andere bestimmte Zeichen anwendet. Der Inhalt von 5 + 7 = 12 is (wenn einer es nicht wüßte) genau das, was den Kindern Schwierigkeiten macht, wenn sie diesen Satz im Rechenunterricht lernen. [PR §103 p. 116]

> If we ask: But what then does '5 + 7 = 12' mean – what kind of significance or point is left for this expression – the answer is, this equation is a rule for signs which specifies which sign is the result of applying a particular operation (addition) to two other particular assigns. The content of 5 + 7 = 12 (supposing someone didn't know it) is precisely what children find difficult when they are learning this proposition in arithmetic lessons. [PR §103 p. 126]

> 'Die Gleichung ergibt a' heißt: Wenn ich die Gleichung nach gewissen Regeln transformiere, erhalte ich, also wie die Gleichung 25 x 25 = 620 besagt, daß ich 620 erhalte, wenn ich auf 25 x 25 die Multiplikationsregeln anwende. [PR §150 p. 165]

> 'The equations yields *a*' means: If I transform the equation in accordance with certain rules, I get *a*, just as the equation 25 x 25 = 620 says that I get 620 if I apply the rules of multiplication to 25 x 25. [PR §150 p. 175]

Mathematical equations are statements of the form $a = b$ where one of the terms '*a*' or '*b*' is a calculation term and the other is a final result term. In consequence, equations of the form '7 = 7' or '3 + 4 = 3 + 4' or the false '3 = 12', where the expressions on both sides of the '=' sign belong to the same grammatical category, are not calculation statements.

Every mathematical equation of the form $a = b$ or '*b* is *a*' expresses that *b* is the final result of calculating *a* (or vice versa). This holds not only of equations expressed with the help of the '=' sign, but also of equations expressed in prose like 'two is the positive square root of four' or 'seven is the least common denominator of twenty one and fifty six'. The particle 'is' does not work as a copula here either. It connects a calculation with its final result. Two is the positive square of four means that two is *the correct final result of calculating* the positive square of four. In general, calculation statements of the form '*a* is *b*' say that *a* is the correct result of calculating *b*. Despite their surface grammar, these are not propositions of the form $F(x)$, where x is an object, and F is a property. The number seven is not an object, and being the least common denominator of twenty-one and fifty-six is not one of its properties. Wittgenstein addresses this in §102 of the *Philosophical Remarks*, where he says that using '=' can make numerical assertions appear to refer to genuine concepts, when they do not.

IV. An Extension of Frege's Context Principle

A. The Context Principle

The Context Principle is a fundamental principle [*Grundsätze*] of Frege's philosophical enquiry. In the introduction to his *Foundations of Arithmetic*, he formulates this principle as "never to ask for the meaning of a word in isolation, but only in the context of a proposition" [*nach der Bedeutung der Wörter muss in Satzzussammenhänge, nicht in ihrer Vereinzelung gefragt werden*].

> But we ought always to keep before our eyes a complete proposition. Only in a proposition have the words really a meaning. It may be that mental pictures float above us all the while, but these need not correspond to the logical elements in the judgement. It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content.[26]

[26]. (Frege 1950, 71)

Wittgenstein thought that the main flaw of Frege's analysis was not considering his own Context Principle seriously enough. Wittgenstein agrees with Frege that words have meaning only in the context of a proposition. However, Wittgenstein also recommended asking for the meaning of a proposition not in isolation but in its larger context of use. For mathematical propositions, this context is their use in calculation. Wittgenstein's extension of Frege's Context Principle commands asking for the meaning of mathematical propositions only in the context of their calculi.

B. Specification and Calculation Propositions and Concepts

According to Frege's analysis, both mathematical *Zahlangaben* like '4 is a number' and equations like '3 + 4 = 7' are propositions of the subject-predicate form F(a), where 'F' is a predicate – referring to a concept – and 'a' is a name – whose referent is an object. Furthermore, since all *Zahlangaben* are arithmetical equations, the concept involved in a *Zahlangabe* is always of the form ' . . . = b' where 'b' is also the name of a number. For Wittgenstein, Frege's distinction overlooks the real difference between these two sorts of mathematical propositions. This difference stems from the different roles they play in their calculi. Both sorts of propositions are rules of mathematical calculi. But they are rules of different sorts. Every calculus has two different sorts of propositions. Propositions of one sort connect calculations with their results. Call these '*calculation propositions*'. Propositions of the second sort specify the calculus' elements. Call propositions of this sort '*specification propositions*'. In elementary arithmetic, the calculation propositions are the equations. Mathematical equations connect calculations with their final results. For example, an arithmetical equation like '3 + 4 = 7' says that adding three plus four results in eight. A mathematical proposition like '7 is a number' does not. It says that '7' belongs to the category of number. 'Being a number' is not a calculation, but a calculus category. A category's elements share the same role in the calculus. Saying that 7 is a number specifies the role of '7' in the calculus. Frege overlooks this difference as well, while Wittgenstein assigns it a central role in his philosophy of mathematics.[27]

A previous section showed how most arithmetical terms in a calculus are either calculation or result terms. However, not all mathematical terms fit into these categories. Consider the term 'number'. Even though basic in arithmetic, 'number' is neither a result nor a calculation term. It does not occur in either side of the '=' sign in arithmetical equations.[28] Other examples of arithmetical terms of this sort are 'addition', 'unit', 'equation', etc. They also describe calculation and result terms internally, but they are not calculation or result terms themselves. They occur in internal descriptions not corresponding to any calculation whatsoever. Call these descriptions '*specification propositions*'.

The distinction between calculation propositions and specification propositions corresponds to a distinction at the level of mathematical concepts. The concepts that occur in specification propositions differ from those in calculation propositions. On a superficial level, statements of the subject + predicate form express propositions of both sorts. However, the concepts that occur as predicates in the specification propositions do not occur as subjects or predicates in calculation statements.

In the case of elementary arithmetic, 'number', 'addition', 'equation', etc. are specification concepts. In consequence, '4 is a number' and '3 + 4 is an addition' are specification propositions. They differ from calculation propositions in that they do not describe the result of any calculation in the calculus they specify. They can play the role of calculation propositions in other calculi, but not in the calculus they specify. For example,

27. This distinction is essential for discussing Wittgenstein's account of mathematical modality.

28. It certainly occurs in arithmetical statements, but its role is singular. It expresses equations in prose. The statement '7 is the sum of *numbers* 3 and 4' expresses the equation '3 + 4 = 7'. However, in these cases, it still does not express a calculation or its result.

constructing a calculus where propositions like ‘4 is a number’ or ‘3 + 4 = 7 is an equation’ are calculation propositions is possible. Wittgenstein found that much of the logicists’ work on the foundations of arithmetic is of this sort. What Frege achieved by giving a formal definition of number was a new calculus in which propositions like ‘4 is a number’ are calculation propositions. He has constructed a calculus where the concept of number is not a specification one, but a calculation one. Hence, it makes sense to say in Frege’s framework ‘I calculate whether 4 is a number’.

In summary, mathematical propositions may be either calculation or specification propositions. Calculation propositions connect calculations with their results. Specification propositions distribute the elements of the calculus into categories.

V. Beyond Arithmetic, Proofs as Calculations

Following Wittgenstein’s own presentation, this chapter has focused mainly on arithmetic. However, Wittgenstein’s notion of calculation covers other mathematical processes, like counting, drawing geometrical figures, and proving theorems within a formal system. Hence, his theory of calculation constitutes a general philosophy of mathematics. Consider, for example, the important case of proving a theorem within a formal system. According to Wittgenstein, proving a theorem within a system is a calculation just like adding or multiplying. The only difference is that its final result is a mathematical proposition, instead of a number.

Compare the grammatical analysis of calculation to proving theorems. The transitive uses of the verbs ‘to calculate’ and ‘to prove that’ in mathematics differ in the expressions they accept as direct complements. Their direct complements are grammatically different. In the case of ‘to calculate’, the direct object is a nominal phrase. However, this is not the case for the verb ‘to prove that’. Unlike the verb ‘to calculate’, the verb ‘to prove that’ does not accept calculation terms, like ‘3+4’ or ‘the supremum of set E’, as their direct complements. It does not accept result terms either. Its direct compliment is a mathematical statement. The only exception is descriptive nominal phrases which refer to mathematical statements. In mathematics, a mathematical statement often follows the expression ‘to prove that’. In other words, the direct object is a mathematical statement.[29] The proof’s object is a mathematicalstatement.[30] Genuine propositions are unprovable. It makes sense to talk about proving that 3 + 4 = 7 or that every equation of second degree has two roots. However, it does not make sense to ask for the proof that there are twenty-four pages in this chapter or fifty states in the Union. The latter propositions are genuine ones, not mathematical. Only mathematical propositions are provable.[31] In consequence, mathematical propositions are themselves result terms corresponding to the calculation of proof. Wittgenstein’s analysis of calculation applies to them as well.

At first sight, calculating in arithmetic and proving theorems in a formal system seem to be very different, because most mathematical calculations are operations. They have one and only one correct final result. However, the concept ‘being a theorem of a system’ usually covers a multiplicity of propositions. While 25 x 25 has only one product, there are many theorems of PA. In consequence, proving a theorem seems to be not a calculation. Calculations are univocal processes. Nevertheless, this does not present a problem for Wittgenstein’s theory. For him, ‘being a theorem’ is not a calculation concept. It is a

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29. Even in cases where the direct object is not a formula, the non-symbolic statement also expresses a mathematical proposition. Wittgenstein scorns mathematical prose, because expressions with non-formal meaning are confusing. For a statement to be mathematical, the meaning of all its parts must be purely mathematical. Every word must make sense only inside the system of calculation. Mathematical propositions cannot say anything else, and to phrase them in a way that may suggest otherwise is philosophically misleading.

30. Mathematical propositions of the form ‘$\vdash p$’ or ‘p is a theorem’ are mathematical, because they express that ‘p’ is the result of the calculation of proving it as a theorem.

31. Cf. Chapter 4 for a detailed account of the role mathematics plays in proofs of non-mathematical propositions.

specification concept, like ‘product’ or ‘root’. Mathematical expressions of the form ‘$| p$’ or ‘*p* is a theorem’, are not calculation propositions, but specification ones. Just as many numbers are roots or products, so many propositions are theorems. Furthermore, just as saying that one number is a product means that it is the final result of some multiplication, a proposition being a theorem means that it results from a proof. In this sense, the proof is the calculation. Consequently, the calculation propositions of this calculus are the units of theorems and their proofs.

> Man könnte auch so sagen: Der völlig analysierte mathematische Satz ist sein eigener Beweis.
> Oder auch so: der mathematische Satz ist nur die unmittelbar sichtbare Oberfläche des ganzen Beweiskörpers, den sie vorne begrenzt.
> Der mathematische Satz ist – Im Gegensatz zu einem eigentlichen Satze – *wesentlich* das letzte Glied einer Demonstration, die ihn als richtig oder unrichtig sichtbar macht [PR §162 p. 182]
>
> We might also put it like this: the completely analysed mathematical proposition is its own proof.
> Or like this: a mathematical proposition is only the immediately visible surface of a whole body of proof and this surface is the boundary facing us.
> A mathematical proposition – Unlike a genuine proposition – is *essentially* the last link in a demonstration that renders it visibly right or wrong. [PR §162 p. 192] [32]

Just as every calculation has a unique result, mathematical propositions have unique proofs. Mathematical theorems with different proofs are different propositions. For Wittgenstein, calculations and proofs are rule-governed activities. They have one single and unique result. A calculation failing to arrive at any result is not a calculation, but an idle, aimless wander. Talking about calculations with more than one result is nonsense too. The obvious case of

32. “A mathematical proposition is related to its proof as the outer surface of a body is to the body itself. We might talk of the body of proof belonging to the proposition. Only on the assumption that there’s a body behind the surface, has the proposition any significance for us. [PR §162 p. 192] Here again, one can only say: look at the proof, and you will see *what* is proved here, what gets called “the proposition proved.” [PG Pt. II §13 p. 301] [I have] likened the conclusion of a mathematical proof to the end surface of a cylinder. The proved proposition is the end surface of the proof, *a part of it*. . . But in mathematics the proof is not a symptom, because the proved proposition is part of the proof. [WL *Philosophy for Mathematicians* 1932-33 §10 p. 221]

equations with more than one root is not a counter example, because it actually hides a multiplicity of mathematical propositions. The process of arriving at each equation's root is different. Each one is a different calculation. Furthermore, the process of arriving at the knowledge that they are both roots of the equation is yet another calculation. In either case, the connection between process and result is so intimate that they cannot be understood in isolation. They provide meaning for each other.

VI. Conclusion

This chapter expounds on the essential notions in Wittgenstein's claim that mathematical propositions are ultimately grammatical. Throughout the *Big Typescript*, *Philosophical Grammar* and *Philosophical Remarks,* Wittgenstein develops his ideas on mathematics largely through examples from elementary arithmetic. Following his own presentation, this chapter begins exploring Wittgenstein's ideas through his analysis of numerical expressions.

Wittgenstein developed his ideas in response to Frege's seminal work on the concept of number. Wittgenstein shares two substantial, methodological principles with Frege: the Grammatical Principle and the Context Principle. For Frege, as well as Wittgenstein, grammatical distinctions have strong philosophical significance. Both believe that every significant philosophical distinction is already present in the grammar of language. For them, the meaning of two expressions *is* logically different if and only if their replacement affects the grammar of the expression in which they occur.

Frege and Wittgenstein also share a strong faith in the philosophical importance of context. For Frege, words have meaning only in the context of propositions. Understanding the meaning of a word requires analyzing its role in complete sentences. Wittgenstein extends Frege's Context Principle to cover sentences as well as words. For him, understanding the meaning of a sentence requires an analysis of its role in a larger system of propositions (or in other sentences), too. In particular, understanding the meaning of numerical statements [*Zahlangaben*] requires an analysis of their roles in the contexts of their use. In the case of mathematical *Zahlangaben*, it requires understanding their roles in calculation. In the case of non-mathematical ones, it requires an analysis of their role in the application of mathematics. Both analyses are essential for the full understanding of numbers in mathematics. If mathematical propositions are grammatical, this would manifest itself in their roles in both calculation and application. The next two chapters follow Wittgenstein's investigation of mathematical calculation and application. The results from both investigations paint a full picture of the grammatical role of mathematical *Zahlangaben* in particular, and mathematical propositions in general.