Chapter 3

The Grammar of Calculation

I. Introduction

Die Mathematik besteht ganz aus Rechnung. PG §40 p.924	Mathematics consists entirely of calculation. PG §40 p. 468

The previous chapter introduced Wittgenstein's thesis that mathematical propositions connect calculations with their final results. Understanding why this also means that the mathematical proposition is a grammatical rule of its calculus requires conceiving of calculation as a rule-governed linguistic practice and of calculi as grammatical systems. This chapter addresses calculation in Wittgenstein's grammatical account of mathematics. Wittgenstein claims that the final result of calculation is correct only if a correct performance of the calculation produces it. Every mathematical calculus is a linguistic system with its own grammar. This grammar provides the criteria for correct and incorrect results. Wittgenstein bases his vision of mathematics as grammar on the claim that all mathematical problems – problems solvable by counting, calculating, drawing a geometrical figure, or proving a theorem in a formal system — are ultimately problems of deciding whether or not an expression fits in a mathematical category. For the most part, this argument appears in Part II Section V of *Philosophical Grammar*, and Section XIII of the second part of the *Philosophical Remarks*. Wittgenstein contrasts his account of calculation with the more traditional Platonistic account, where calculations are *explorations* into some yet unmapped, mathematical territory. Wittgenstein views calculations more as rule-governed *searches* over well-defined grammatical spaces.

II. Mathematics as Calculation

A. "There's a *Fascination* here:" Mathematical Problems and Problems of Mathematical Investigation

Das Unglück ist, daß unsere Sprache so grundverschiedene Dinge mit jedem der Worte 'Frage', 'Problem', 'Untersuchung', 'Entdeckung' bezeichnet.	The sad thing is that our language uses each of the words 'question', 'problem', 'investigation', 'discovery' to refer to such fundamentally different things.
PG §22 p.706	PG §22 p. 359

At §22, the very beginning of Section V of the *Philosophical Grammar*'s second part, Wittgenstein distinguishes between tasks of calculation [*Aufgaben der Rechnung*] (also called 'mathematical problems' [*mathematische Probleme*] or 'problems *in* mathematics' [*Probleme in der Mathematik*]) – and problems of mathematical investigation [*Problem der mathematischen Forschung*] (also called 'problems *of* mathematics' [*Probleme der Mathematik*]), such as Fermat's Last Theorem or Riemann's Hypothesis. There, he writes:

> 'Wenn auf die Lösung – etwa – des Fermat'schen Problems Preise ausgesetzt sind, so könnte man mir vorhalten: Wie kannst Du sagen, daß es dieses Problem wohl nicht geben. Ich müßte sagen: Gewiß, nur mißverstehen die, die darüber reden, die Grammatik des wortes "mathematisches Problem" und des Wortes "Lösung". Der Preis is eigentlich auf die Lösung einer naturwissenschaftlichen Aufgabe gesetzt; auf das *Äußere* der Lösung (darum spricht man z. B. auch von einer Riemann'schen *Hypothese*). Die Bedingungen der Aufgabe sind äußerliche; und wenn die Aufgabe gelöst ist, so entspricht, was geschehen ist, der Stellung der Aufgabe, wie die Lösung einer physicalischen Aufgabe dieser Aufgabe. [p. 712]
>
> Suppose prizes are offered for the solution – say – of Fermat's problem. Someone might object to me: How can you say that this problem doesn't exist? If prizes are offered for the solution, then surely the problem must exist. I would have to say: Certainly, but the people who talk about it don't understand the grammar of the expression "mathematical problem" or the word "solution". The prize is really offered for the solution of a scientific problem; for the *outside* of the solution (hence also for instance we talk about Riemann's *Hypothesis*). The conditions of the problem are external conditions; and when the problem is solved, what happens corresponds to the setting of the problem in the way in which solutions correspond to problems in physics [p. 362]

Under the vocabulary developed in the first chapter, a mathematical problem always asks if a calculation proposition is correct. Every mathematical problem tests the correctness of a calculation proposition. A problem is mathematical only if it can be formulated as a question ‘Is *p* so?’, where *p* is a calculation proposition.

> The mathematician’s activity is carried on in a particular sphere. A question is part of a calculus. What does it prompt you to do? [WL *Philosophy for Mathematicians* 1932-33 §10 p. 222]

> Nur dort kann man in der Mathematik fragen (oder vermuten), wo die Antwort lautet: ‘Ich muß es ausrechnen’. [PR §151 p. 165]

> We may only put a question in mathematics (or make a conjecture), where the answer runs: ‘I must work it out’. [PR §151 p. 175]

A mathematical proposition is true in a completely different sense than other kinds of propositions, whether they are specification propositions or non-mathematical ones, are true. In the case of a calculation proposition *p*, *p* is true means that *p* is correct, according to the calculus. Since a calculation proposition connects a calculation to its result, a calculation proposition is correct if and only if it connects a calculation with its correct result. The calculation ultimately bears the solution of a mathematical problem.

The boundaries between mathematical problems and problems of mathematical investigation are sometimes difficult to see. In §23, Wittgenstein examines whether or not the equation ‘$x^2 + ax + b = 0$’ has a solution over the real numbers. Whether or not any real number satisfies the equation seems to be not a matter of mere calculation, but an ontological question about the existence of an abstract object with a certain property. However, it is, in fact, a mathematical problem equivalent to the question ‘(n) $n^2 + an + b = 0$?’ In consequence, calculating its solution is possible. The prose of the original question obscured its mathematical nature. In mathematical prose, the expressions ‘there is’ and ‘exists’ correspond to the formal symbol ‘ ’.[1] Outside mathematics, however, they mean something completely different.

> In Wirklichkeit ist Existenz das, was man mit *dem* beweist, was man “Existenzbeweis” nennt. . . Wir haben keinen Begriff der Existenz unabhängig von unserm Begriff des Existenzbeweises. [PG PT. II §24 p. 736]
>
> Really, existence is what is proved by what we call “existence proof”. . . We have no concept of existence independent of our concept of existence proof. [PG PT. II §24 p. 374]

Problems of mathematical investigation are not problems of mathematical calculation. They can be formulated as questions ‘Is *p* so?’. However, here, *p* is not a calculation proposition. Instead, *p* is a genuine *hypothesis*. Problems of mathematical investigation seem to be a call for calculating what cannot be calculated. Mathematical problems are tasks of calculation that demonstrate the correctness of calculation propositions. By contrast, problems of mathematical investigation are pseudo-mathematical problems. Problems of mathematical investigation are of two kinds: (1) genuine problems disguised as mathematical problems or (2) ungrammatical pseudo-problems. In the first case, they pose external questions about the calculi. Since “calculus is not a mathematical concept” [*Kalkül ist kein mathematischer Begriff* PG Pt. II §12 p. 580 (p. 296)], calculations cannot say anything about their own calculi. No *metacalculi* exist. For example, the question if a given calculus has any external application is an empirical question disguised as a mathematical one. The calculus itself cannot answer this question. “That will show itself soon” [*das wird sich dann schon zeigen* PG Pt. II §15 p. 600 (p. 306)] is the only possible answer. Applying the calculus is the only way to demonstrate its applicability. So-called mathematical conjectures [*Vermutungen*] of this kind are actually physical hypotheses, not mathematical ones.

1. PG Pt. II §24

Other problems of mathematical investigation are not-yet-mathematical problems. These questions are strings in mathematical language whose surface grammar is similar to that of calculation propositions. However, they cannot be calculated yet because they do not fit in any existent calculus. They do not amount to actual mathematical problems, because their calculation does not belong to any current calculus. Fermat's Last Theorem is a problem of this sort. The symbols in the expression $x^n + y^n = z^n?$ certainly belong to a mathematical calculus. However, their combination does not, yet.

> Ich behaupte ja nicht: wenn jemand sich mit dem Fermatschen Problem beschäftigt, so ist das falsch oder unberechtigt. Durchaus nicht! Wenn ich z. B. eine Methode habe, um nach ganzen Zahlen zu suchen, welche die Gleichung $x^2 + y^2 = z^2$ erfullen, so kann mich die Fomrrl $x^n + y^n = z^n$ anregen. Ich kann mich von einer Formel anregen lassen. Ich werde also sagen: hier liegt eine *Anregung* vor, aber keine *Frage*. Die mathematische 'Probleme' sind immer solche Anregungen. Diese Anregungen sind nicht etwa eine Vorbereitung auf einen Kalkül. [PR Appendix II, p. 321]

> I'm certainly not saying: if anyone concerns himself with Fermat's Last Theorem, that's wrong or illegitimate. Not at all! If, for instance, I have a method for looking for whole numbers satisfying the equation $x^2 + y^2 = z^2$, the formula $x^n + y^n = z^n$ can disturb me. I can allow myself to be disturbed by a formula. And so I shall say: there's a *fascination* here but not a *question*. Mathematical 'problems' always stir us up like this. This kind of fascination is not really the preparation of a calculus. [PR Appendix II, p. 334]

> Der Fermatische Satz hat also keinen *Sinn*, solange ich nach der Auflössung der Gleichung durch Kardinalzahlen nicht *suchen* kann. [PR §150 p. 165]

> Thus Fermat's Theorem makes no *sense* until I can *search* for a solution to the equation in the cardinal numbers. [PR §150 p. 175]

Solving this kind of problem requires the construction of a new calculus where that combination of signs makes sense. It is necessary to construct a calculus where a calculation answers the question in the old problem. Only then, the question becomes meaningful and a proper mathematical problem. However, then, calculation does not solve the original pseudo-problem of mathematics. It solves the new mathematical one.

In the end, Wittgenstein finds that the only thing these two different sorts of problems have in common is that they have a solution.[2] However, only one of them has a method.

B. Method and Solution

The mathematical processes involved in the solution of mathematical problems are rule-governed activities. Calculating, proving theorems within a formal system, *et cetera* are all activities governed by rules. These rules provide the criteria determining whether their results –and, hence, the solution to the mathematical problem – are correct or incorrect. For example, the question '25 x 25 = 625?' is mathematical only if calculation can solve it. If a general method of solution exists, the question is mathematical. An addition is a mathematical problem if its calculation follows a set of rules. The correct answer to the question '25 x 25 = 625?' results from following the rules throughout the calculation. In §25, Wittgenstein writes:

> "Du sagst 'Wo eine Frage ist, da ist auch ein Weg zu ihren Beanwortung', Aber in der Mathematik gibt es doch Fragen, zu daren Beanwortung wir keinen Weg sehen." – Ganz richtig, und daraus folgt nur, daß wir in diesem Fall das Wort 'Frage' in anderem Sinn gebrauchen, als im oberen Fall. Und ich hätte vielleicht sagen sollen "es sind hier zwei verschiedene Formen und nur für die erste möchte ich das Wort 'Frage' gebrauchen." Aber dieses Letztere ist nebensächlich. Wichtig ist, daß wir es hier mit zwei verschiedenen Formen zu tun haben. (Und daß Du Dich in der Grammatik des Wortes 'Art' nicht auskennst, wenn Du nun sagen willst, es seien eben nur zwei verschiedenen *Arten* von Fragen.) [PG §25 p. 748]

> "You say 'Where there is a question, there is also a way to answer it', but in mathematics there are questions that we do not see any way to answer." Quite right, and all that follows from that is that in this case we are not using the word 'question' in the same sense as above. And perhaps I should have said "Here there are two different forms and I want to use the word 'question' only for the first". But this latter point is a side-issue. What is important is that we are here concerned with two different forms. (And if

2. PR §151

> you want to say they are just two different *kinds* of question you do not know your way about the grammar of the word "kind.") [PR 25 p. 380]
>
> If a question is asked for which there does not exist a method of solution, does the question have meaning? I have said "No." [WL *Philosophy for Mathematicians* 1932-33 §10 p. 221]

At the beginning of Section 23 of *Philosophical Grammar*, Wittgenstein states that the truth of a calculation proposition is not discovered –or invented for that matter, but *checked*. For Wittgenstein, systems of propositions define methods of solution. The system of propositions contains those propositions which are correct according to it. In consequence, a proposition expresses a correct calculation if it is one of the rules defining the calculation. Checking the truth of a calculation proposition, is looking for it among the propositions defining the general method of solution.

C. Calculations are Transitions between Expressions According to a Rule

Ich will also sagen: das Arithmetische ist nicht der Anlaß, *5* und *7* zusammenzugeben, sondern der Vorgangg und was dabei herauskommt. PR §104	I want to thus say: arithmetic is not the cause to combine 5 and 7 but the process and its outcome. PR §104

Calculations are not events. They are rule-governed practices. This distinction lies at the heart of Wittgenstein's discussion between 'following' and 'inferring' at the beginning of the *Remarks on the Foundations of Mathematics*. For Wittgenstein, inferring is an activity. It is something people do. People infer some propositions from others. Whether or not these propositions follow from each other is a very different matter. That a proposition follows from another does not require anyone actually inferring it. Logic is not natural history. That a proposition *A* follows from *B* does not mean that we — whoever *we* are — *usually* infer *A* from *B*. It relies on the logical rules of inference. In the case of logic or mathematics, 'to follow' is a grammatical relationship between mathematical propositions.

> Das kann auf dem Papier, mündlich, oder 'im Kopf' vor sich gehen. - Der Schluß kann aber auch so gezogen werden, daß der eine Satz, ohne Überleitung, nach dem andern ausgesprochen wird; oder die Überleitung besteht nur darin, daß wir "Also", oder "Daraus folgt" sagen, oder dergleiche [RFM §6 p. 39].

> This [inferring] may go on paper, orally or 'in the head'. —The conclusion may however also be drawn in such a way that the one proposition is uttered after the other, without any such process; or the process may consist merely in our saying "Therefore" or "It follows from this", or something of the kind [RFM §6 p.5].

On the other hand, 'Following' has nothing to do with assertions – i.e. utterances [*Behauptung*] – which are events themselves, but with propositions [*Satz*]. The 'following' of one mathematical proposition to another is a transition. However, it is not a process or event. Following does not happen in time.

> Wenn wir sagen: "dieser Satz folgt aus jenem", so ist hier "folgen" wieder *unzeitlich* gebraucht. (Und das zeigt, daß dieser Satz nicht das Resultat eines Experiments ausspricht.) [RFM §103 p.30e]

> When we say: "This proposition follows from that one" here again "to follow" is being used *non-temporally.* (And this shows that the propositions does not express the result of an experiment.) [RFM §103 p.30]

Unlike drawing conclusions in everyday life, logical inference results from calculation. Wittgenstein defines inference as "the derivation of one sentence [*Satz-zeichen*] from another according to a rule."[3] Derivation is a logical calculation whose result is a logical inference. The forming [*Bilden*] of the derivation is "the comparison of both sentential expressions [*Satz-zeichen*] with some paradigm or another which represents the schema of the transition." Consider the derivation of the sentence '(PvQ)' from '(PvQ)&R', according to the rule of conjunction elimination. The derivation's first step is to take a scheme *representing* the rule:

3. RFM §6

$$\frac{A\&B}{A}$$

Second, compare the two sentences to those in the paradigm.[4] Wittgenstein recommends arranging them in the form of the scheme:

$$\frac{(\mathrm{PvQ})\&\mathrm{R}}{(\mathrm{PvQ})}$$

Comparing the signs in the calculation with those in the scheme allows for the formation of the derivation. Recognizing the adequate correspondence justifies the correctness of the derivation. More complex derivations require the reiteration of these simple steps.

All calculations follow this simple process. Every calculation is a transition from one expression to another according to a rule. The 'forming' of every step in the calculation involves the comparison of expressions with some paradigm representing the schema of the transition.[5]

III. Mathematical Correctness

Man kann Mathematik nicht schreiben, sondern nur machen. PR §157	You can't write mathematics, you can only do it. PR §157

A calculation proposition and other kinds of propositions – specification propositions and non-mathematical ones – are true in completely different senses. In the case of a calculation proposition *p*, that *p* is true means that *p* is the correct result of a calculation performed in accordance with the rules of the calculus. The calculation always bears the solution of a mathematical problem. In consequence, it is more proper to talk about mathematical

4. RFM §31

5. Sometimes the paradigm may not be physically present, but "in the head." For example, having the tables of multiplication at hand is not necessary for multiplying 3 x 4.

correctness than mathematical *truth.* The contrary of a true mathematical proposition is not a false one, but nonsense resulting perhaps from an incorrect calculation.

> Was ist das Gegenteil des Beweisenen? – Dazu muß man auf den Beweis schauen. Man kann sagen: das Gegenteil des beweisenen Satzen ist das, was statt seiner durch einen bestimmten Rechungsfehler im Beweis bewiesen worden wäre. [PG Pt. II §24 p. 732]
>
> What is the contradictory of what is proved? – For that you must look at the proof. We can say that the contradictory of a proved proposition is what would have been proved instead were a particular miscalculation made in the proof. [PG Pt. II §24 p. 372]

Calculations cannot be false, but they can be incorrect.

A. Wittgenstein's Anti-Platonism

1. Platonist Mathematical Explorations

As part of Wittgenstein's criticism of Frege's philosophy of mathematics, Wittgenstein takes an anti-platonist attitude regarding calculations. Frege's position is usually characterized as a kind of platonism. Frege wrote that numbers are "self-subsistent"[6] objects with definite "properties that can be specified."[7] He also wrote that "the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name."[8] Of course, Frege did not originate the idea that mathematics consists of a body of discoveries about an independent reality made up of objects like numbers, sets, shapes and so forth. Furthermore, platonism did not receive its name until 1934, when Paul Bernays baptized it in his lecture 'On Platonism in Mathematics.' He characterized 'Platonism' as the view of mathematical objects "as cut off from all links with

6. G. Frege, *The Foundations of Arithmetic* (Evanston: Northwestern University Press, 1950), §55

7. Ibid. §8

8. Ibid. §96

the reflecting subject." He continued, "Since this tendency asserted itself especially in the philosophy of Plato, allow me to call it 'Platonism'."

Many mathematicians enthusiastically adhere to the ideals of Platonism. From Frege's contemporaries like Emile Borel and Charles Hermite, to more recent thinkers like Kurt Gödel and Roger Penrose, several eminent mathematicians have embraced some form of Platonism. Nevertheless, Platonism is not a unitary doctrine or a definite philosophical theory. It is a cluster of metaphors that together make up a heuristic picture of mathematics. The *Macmillan Encyclopedia of Philosophy*, edited by Paul Edwards[9], offers the following definition of *platonism*,

> By platonism is understood the realist view, akin to that of Plato himself, that abstract entities exist in their own right, independently of human thinking. According to this view number theory is to be regarded as the description of a realm of objective, self-subsistent mathematical objects that are timeless, non-spatial, and non-mental. **Platonism conceives it to be the task of the mathematician to explore this and other realms of being.** Among modern philosophers of mathematics Frege is a pre-eminent representative of platonism, distinguished by his penetrating lucidity and his intransigence.[10]

In "Frege's Influence on Wittgenstein," Erich Reck factors this definition of platonism into three heuristic metaphors:

> (*i*) numbers and other mathematical entities are "abstract objects" which exist "in their own right;" (*ii*) in mathematics we "describe" these objects, i.e., we talk about them as members of a "mathematical realm;" and (*iii*) the task of the mathematicians is to "explore" this realm, i.e., to find out what is "objectively the case" in it.[11]

From his analysis of the work of Paul Benacerraf, Erich Reck concludes that Platonism remains a popular philosophical position for mathematicians, because it has set its agenda on the philosophy of mathematics. For the last hundred years, Platonism has defined the

9. Paul Edwards, ed., *The Encyclopedia of Philosophy* (New York: Macmillan: 1967)

10. (Barker 1967, 529) Emphasis added.

11. Erich Reck, "Frege's Influence on Wittgenstein" in William W. Tait ed., *Early Analytic Philosophy* (Chicago: Open Court, 1997), 126.

four explanatory goals of a philosophical account of mathematics. They are: (*i*) the nature of mathematical entities, (*ii*) the meaning of mathematical expressions, (*iii*) the objectivity of mathematical propositions, and (*iv*) the possibility of mathematical knowledge.[12] Offering an alternative explanation of these four points is not sufficient. A true challenge to Platonism requires redefining the agenda in philosophy of mathematics. Wittgenstein's grammatical picture successfully accomplishes this.

Wittgenstein rejects Platonism's three metaphors. For him, (*i)* mathematical entities are not genuine objects, much the less abstract or self-subsistent. (*ii*) Mathematical propositions are not descriptions. Mathematics does not describe mathematical entities. And (*iii*) mathematical calculations are not explorations of some esoteric 'mathematical realm', but searches in well-defined, logical spaces. However, Wittgenstein's objection goes beyond the mere rejection of these metaphors. It offers a new agenda for the philosophical investigation of mathematics.

Wittgenstein substituted the Platonists' basic questions with his own. The nature of mathematical entities stopped being an ontological issue and became a grammatical one. Since mathematical entities are not objects, but grammatical categories, questions about their independent subsistence become meaningless. Their existence becomes bound to the calculus they belong to. Calculations are not descriptions. Accordingly, questions about their descriptive powers or their truth are largely mistaken. Trying to explain how mathematical statements describe the abstract world of mathematics is nonsense. The relevant philosophical question is 'How do mathematical propositions rule calculations?' As rules, mathematical propositions are neither true or false. As calculations, they are either correct or incorrect. Wittgenstein substitutes for the Platonist's 'truth' of mathematical

12. Ibid. 128.

propositions in terms of their accurate description of mathematical reality, the correctness of calculations.

2. Searches and Explorations

> A mathematician is a blind man in a dark room
> looking for a black cat which isn't there.
> Charles Darwin.

Wittgenstein's objection to the Platonists' account of calculation is the conflict between two metaphors. For Platonists, mathematical calculations are excursions, while Wittgenstein sees them as searches.

> Und 'suchen' muß immer heißen: systematische suchen. Es ist kein Suchen, wenn ich im unendlichen Raum nach einem Goldring umherirre. [PR §150 p. 165]

> And 'search' must always mean: search systematically. Lost and wandering in infinite space looking for a gold ring is not searching. [PR §150 p. 175]

The view of calculation as exploration is closely linked to that of an independent mathematical realm. In geography, a successful exploration consists of the discovery and faithful description of a new territory. Similarly, for platonists, the discovery and faithful description of new mathematical territory makes for mathematical success. Frege expressed this position in his late article "Thoughts" (1918-19),

> Thus for example, the thought we have expressed in the Pythagorean Theorem is timelessly true, true independently of whether anyone takes it to be true. It needs no owner. It is not true only from the time when it is discovered; just as a planet, even before anyone saw it, was in interaction with other planets.[13]

For Wittgenstein, the metaphor of calculations as explorations is misleading, because it suggests that mathematical truth is independent of the calculation. It suggests that the success of a mathematical calculation depends on whether it produces a correct or incorrect result. It suggests that a calculation is correct if it reveals a mathematical truth. Instead, in

[13]. Gottlob Frege, *Collected Papers*, ed. by Brian McGuiness (Oxford: Basil Blackwell, 1984), 363.

mathematics, the truth of the result depends on the success of the calculation. For example, to correctly calculate the product of 25 by 16, arriving at the correct result – 400 – is not sufficient. 400 is the product of 25 by 16, precisely because it is the result of multiplying these numbers, not the other way around. The product of two numbers is simply the result of correctly multiplying them. The calculation is not correct, because it produces the correct result. The result is correct, because the calculation is successful. This holds for the solution of any mathematical problem, for example, proving a theorem in a formal system. A proof is not correct, just because it results in a theorem. Instead, the result of a proof is a theorem, if the proof is correct.

In §150 of the *Philosophical Remarks*, Wittgenstein writes:

> Was das Verständnis erschwert, ist die falsche Auffassung, als wäre die *allgemeine* Lösungsmethode nur ein – nebensächliches – Hilfsmittel zur Erhaltung von Zahlen, die die Gleichung befriedigen. Während sie an sich ein Aufschluß über das Wesen (die Natur) der Gleichung ist. Sie ist – wieder – kein nebensächliches Hilfsmittel zum Finden einer Extension, sondern Selbstzweck. [p. 163]

> What makes understanding so difficult is the misconception of the *general* method of solution as only an –incidental– expedient for deriving numbers satisfying the equation. Whereas it is in itself a clarification of the essence (nature) of the equation. Again, it isn't an incidental device for discovering an extension, it's an end itself. [p. 173]

This does not imply that arithmetical – or any other kind of mathematical – propositions are empirical generalizations. Mathematical propositions do not report on the way people perform calculations. The correctness of a calculation does not depend on the performance circumstances. The correctness of a multiplication, for example, is not contingent on the way one multiplies, but on the rules of multiplication. When Wittgenstein talks about the correctness of a calculation, he is not referring to any action, particular or general. Mathematics does not say anything about what goes on in the mind of mathematicians while calculating. Mathematics is not the psychology or sociology of calculation. It is its grammar. Michael Young explains,

> It would indeed be paradoxical to suppose that we can base *a priori* judgements on the observed outcomes of activities performed on paper or 'in our heads' if the empirical judgements that reflect the observations in question served directly as evidence for the arithmetical judgement that we make. [It seems clear that one performs the calculation, for example '79 + 86 = 165'] by first producing a symbolic representation of the mathematical [calculation] concept 'the sum of 79 and 86', (which we do probably by writing down the numerals '79' and '86', one atop the other or one after the other with an addition '+' sign in between), then performing certain activities in which the symbols figure as elements (writing down a '5' at the bottom of the right-most column, carrying the '1', etc.), and by then noting that the outcome of our activities is the numeral string '165' which we take to represent the number that is the actual sum we have calculated.[14]

Certainly, this activity is not mere tinkering, simply happening to yield a certain outcome. A clear and precise system of rules governs calculation. These rules, moreover, do not just provide recommendations as to how to best perform the activity in question. More specifically, they serve to define the activity itself. The rules of addition, for example, specify what it is to calculate a sum. Failing to follow the rules is failing to calculate. A miscalculated addition is no addition at all. An incorrect calculation is no calculation at all. To note the activity's outcome is not merely to observe a certain string of numerals turning up at the bottom of a column. Rather, it is to notice that the procedure of calculation defined by the rules yields this numeral string as its only and inevitable outcome.

Natural language reflects these circumstances in its talk of 'calculation'. After doing everything just as it ought to be done, it is legitimate to claim having performed *the* calculation. It is possible to distinguish among different performances of the activity or different agents performing the calculation. So long as these performances are properly executed, they all qualify as performances of *the* calculation. A mathematical proposition does not merely express the observation that a certain numerical expression turns up at the

14. Michael Young, "Kant on the Construction of Arithmetical Concepts" *Kant-Studien* 73 (1982), 25.

bottom of the column or following the "=" sign. Instead, it expresses that such a numerical expression turns up as *the* outcome of *the* calculation.

Judging the result of a calculation, however, does involve the judgement of matters external to the calculus. In Michael Young's words, "as these comments suggest, when we ground an arithmetical judgement on the performance of a calculation we do, in fact, pre-suppose the truth of a number of empirical judgements."[15] For example, arithmetical calculation requires successfully recognizing the symbols in the calculation as belonging to their proper number-types. However, these judgements are not part of the content of mathematical propositions. They do not provide evidence for the truth of the mathematical proposition. The role these judgements play is simply supporting the judgement that one has performed the calculation correctly and that the numerical expression generated is, therefore, *the* outcome of *the* calculation in question.

Making mistakes in these matters is possible, of course. It is possible to misidentify a certain character as a '6', when, in fact, it is an '8'. Accidently skipping a numeral, carrying improperly, and so on, are all possible mistakes. In these cases, the numerical expression appearing at the end of the performance of the activity may not be the correct one. However, the calculation has not yielded the wrong answer. Correct calculations do not yield wrong answers. Rather, sometimes, being mistaken in thinking to have correctly performed the relevant calculation is possible. However, no arithmetical equation can express this fact. Only an appropriate genuine proposition can. 'I have performed the addition of 64 and 78, and the result was 142' is a genuine proposition. It does not amount to the mathematical proposition '64 + 78 = 142'. The first one may be false. It is possible to miscalculate the addition. The second one, by contrast, is necessary. Furthermore, it

15. Ibid. 24.

determines the truth (or falsity) of the previous one, not the other way around. Attempting to add 64 and 78, but producing a numeral ‘168’ as outcome, the miscalculation is clear, precisely because 64 + 78 = 142.

In committing a mathematical error, or making a mathematical conjecture, the object of false or hypothetical belief is not a mathematical proposition, but a genuine proposition reporting on the (mis)calculation.[16] It is possible to believe that ‘625 + 1223 = 1818’ is true after miscalculating, for example. But then the proposition ‘625 + 1223 = 1818' occurs *de dicto*, not *de re*. The mistaken belief is not about some matters in the abstract mathematical realm. It is about the expression ‘625 + 1223 = 1818’. Namely, it is the belief that it results from a correct calculation. It is not the belief that it expresses a true mathematical proposition.

Just as mathematical success is the success of calculation, mathematical mistakes are the result of miscalculation. In consequence, mathematical calculations, including proofs, are not made of propositions, but of expressions. Their correctness is purely grammatical.

B. Correct Calculations and Correct Results

1. Calculations as Transitions

> Aber *entshuldigen* Sie! . . .
> Im Kalkül interessiert man sich ja immer dafür, was herauskommt: wie seltsam? Das kommt da heraus – und dort das! Wer hätte das gedacht!
> PR Appendix II. p. 319

> But *excuse* me! . . . In the calculus, we are always interested in the result. How strange! This comes out here – and that there! Who would have thought it?
> PR. Appendix II. p. 332

Wittgenstein established that calculation solves all mathematical problems. However, the correctness of the solution and the correctness of the calculation are not equivalent. For any

[16]. The object of false belief is its ‘explicitly grammatical counterpart’ as defined in chapter 7.

mathematical question, a correct calculation always gives the correct solution. However, it is also possible for incorrect calculations to yield the right solution to the problem, perhaps by accident. If the calculation is correct, then the solution is correct. But, in many cases, giving the correct solution does not guarantee the calculation's correctness.

The correct performance of an algorithmic mathematical calculation entails the correctness of its result. A correct calculation always yields the correct result. A well performed multiplication produces the correct product of its factors as result. The result of drawing a circle correctly must be a circle. Also, proving a formula correctly fully justifies calling it a theorem. However, the converse is not true. Sometimes, incorrect calculations also yield correct results. Consider the aforementioned examples of calculating the product of two numbers, drawing a circle and proving a theorem in a formal system. In these cases, a correct result does not always entail a correct calculation. For a proof to be correct, it takes more than the result to be a theorem. An incorrect multiplication may also produce the correct product. It is, in principle, possible to draw a circle free hand. In all these cases, it is possible to obtain the correct result without performing the correct calculation.

However, for Wittgenstein, the calculation entirely determines the correctness of the solution. Wittgenstein insists that calculations are transitions between expressions. He is never clear as to whether the transition happens from the calculation expression to the solution expression or from the expression of the mathematical question to the complete mathematical proposition. Consider the mathematical question '37 x 48 = ?'. Its solution is '745'. Wittgenstein does not say whether the multiplication is the transition from '37 x 48' to '745' or from '37 x 48 = ?' to '37 x 48 = 745'. In either case, the important point is that the multiplication is a transition between both expressions. It connects the calculation with its result. This is what the mathematical proposition expresses. '37 x 48 = 745' expresses this transition. In contrast, finding the correct solution to the problem without calculating does not provide this connection.

The importance of the connection between calculation and result increases with the complexity of the calculation. In the basic cases, the correctness of result and calculation is the same. More complex cases require more than correct results. In the case of arithmetical multiplication, for example, multiplying small integers correctly involves nothing but getting the correct result. That is why children memorize the multiplication tables. Also, multiplying larger integers requires obedience to the rules of multiplication. To know that the calculation is correct requires more than a correct result. It requires something ensuring that the result was actually calculated, instead of guessed or copied. Multiplying small integers by memory and multiplying larger ones by algorithm are different sorts of calculations. Algorithmic calculations may be wrong and still yield correct results. Elementary school math students all over the world have complained about this for years. The correction of the calculation requires something else. It requires a sign or trace evidencing that some calculation yielded the result.

2. Traces

In some cases, the result of a process is not a material object, but a transition. In the case of processes like cooking or building, even when language is ambiguous, telling process from result is relatively easy. Man or nature may perform processes well or poorly, fast or slowly. Their results, as material objects, have material properties which the processes do not. A building may be tall or ugly, modern or old, etc. A food dish may be salty or sweet, well-done or overcooked, etc. However, not all processes produce a material object as result. Consider, for example, the process of growth. The result of growing is not an object, but a transition from one state to another. Examples of processes whose result is a transition are many. The result of movements, transformations, and similar processes is not an object, but a transition. These cases have the extra difficulty that they do not only have a result, but also a *final result*. The final result of these processes is not the transition, but its final state. Consider the growth process that happened in my teen years. In those years I grew from 1.50 m to 1.78 m. The transition from 1.50 m to 1.78 m was the result of this process. The final result, on the other hand, was my being 1.78 m tall.[17]

Mathematical calculations are activities of this sort. Calculation is a process more like going or growing, than cooking or building. The result is not a material object, but a transition. The result of going from one place to another is the transition between being in one place and being in the other. The final result may be being at the arriving place, but the result of going there is the transition from the starting place. The absence of a material object as result does not imply that transitions do not produce anything material. In most cases, transitions leave a material trace. Consider, for example, walking from one edge of a beach to the other. The transition from one edge of the beach to the other is immaterial, but it leaves some marks behind. Since the sand was soft, the walking process could have left certain traces on it. Let us call these material byproducts the 'trace' of the process. As its name indicates, 'traces' trace the result back to the process that produced it. By the trace of someone's steps on the grass, it is possible to connect transition with result, and starting point with the end. This trace shows that a person got to one side of the knoll from the other (result) by walking on the grass (transition). One also knows where exactly that person went (final result). Furthermore, it shows each step from one side to the other.

> What counts in mathematics is what is written down. Symbols obviously interest even the intuititonist, who says that mathematics is not a science about symbols but about meanings – just as a zoologist might say, analogously, that zoology is not a science about the word "lion" but about

17. Ordinary language commonly presents the final result in such a way that it becomes explicit that it is the final result of a transition. In presenting the final result, one uses verbs like 'becoming,' 'getting,' and others lexical indicators like 'now,' 'after that,' which denote transition. I would say that, as a result of growing up during my teenage years, I became 1.78 m or that, as a result, I am 1.78 m now.

> lions. But there is no analogy between mathematics and zoology in this respect. [WL *Philosophy for Mathematicians* 1932-33 §11 p. 225]

In the case of calculations, the trace is the group of symbols written while performing the calculation. For example, when adding 345 and 786, the trace may be something like this:

```
  11
  345
 +786
 1131
```

The result is 1131, but the trace is the complete group of numerals, lines and 'carries' written above. They show not only the calculation performed (the addition of 345 and 786) and its final result (1131), but also the intermediate steps taken. The '1' above the '4' in '345,' says that adding five and six first resulted in a number greater than nine, but less than twenty. The final result is not, properly speaking, part of the trace. Even though the numeral '1131' occurs in the trace of the addition, the actual final result is not a numeral, but a number: 1131.[18]

IV. Conclusion

Wittgenstein distinguishes between mathematical problems and problems of mathematical investigation. Mathematical problems are tasks of calculation that demonstrate the correctness of calculation propositions. By contrast, problems of mathematical investigation are misguided attempts at calculating what cannot be calculated. They are either attempts to calculate non-mathematical propositions or ungrammatical pseudo-propositions.

Every mathematical problem asks if a calculation proposition is correct. A calculation can solve a mathematical problem, if a general method of solution exists. Calculations are not explorations of some esoteric 'mathematical realm', but searches in

18. In these cases, it is feasible to say that the result is the *meaning* of the trace, and the final result is the *meaning* of only a part of it.

well-defined, logical spaces. Checking the truth of a calculation proposition is looking for it among the rules defining the calculation.

Mathematical propositions are true in a completely different sense than other propositions. In the case of a calculation proposition *p*, *p* is true means that *p* is correct, according to the calculus. Since a calculation proposition connects a calculation to its result, a calculation proposition is correct if and only if it connects a calculation with its correct result. The calculation ultimately bears the solution of a mathematical problem. In consequence, it is more proper to talk about mathematical *correctness* than mathematical *truth.* Truth and falsity do not apply to mathematics, only correctness and incorrectness. Mathematical truth is not independent of calculation. A calculation is not correct, because it reveals a mathematical truth. Instead, mathematical truth is the correctness of calculation.

Calculations are not events. Mathematics is not a natural science. It is not the psychology or sociology of calculation. Mathematical propositions are not empirical generalizations. They do not report on the way people perform calculations. A mathematical proposition does not merely express the observation that a certain expression turns up at the end of the calculation. Instead, it expresses that such expression turns up as *the* outcome of *the* calculation. It connects the calculation with its result. In consequence, the solution to a mathematical problem involves more than just producing a correct result. It requires a connection between problem and solution. It requires something ensuring that the result was actually calculated. A sign, tracing the result back to the calculation that produced it, is also necessary.

Every calculation is a transition from one expression to another, according to a rule. The ‘forming’ of every step in the calculation involves the comparison of expressions with some paradigm representing the scheme of the transition.