Chapter 4
Mathematical Application [*Anwendung*]

I. Introduction

Section III of the second part of the *Philosophical Grammar*, 'Foundations of Mathematics', [*Grundlagen der Mathematik*] focuses on the notion of application [*Anwendung*]. As the section's title suggests, Wittgenstein primarily explores the role application plays in the foundations of mathematics. The application of mathematics requires no foundation. Using formal tools for preparing mathematics for its application (proofs of relevance, proofs of consistency, and formal interpretations) is superfluous and misguided. Such efforts are superfluous, because preparing a calculus for application is unnecessary. The application "takes care of itself." They are misguided, because calculation can only solve mathematical problems, and a calculus' applicability is not a mathematical problem. Instead, the quest for a foundation for mathematics is a philosophical problem. Calculation cannot yield a foundation for mathematics, because mathematics is "well enough grounded in itself" [*genug in sich selbst begründet.* PG §15 p. 600 (p. 306)].

Calculations cannot solve anything but mathematical problems. In consequence, mathematical calculations can only solve mathematical problems. Even though mathematical calculations play a central role in the solution of practical problems, they do not offer, entail or justify predictions about affairs outside the calculus. Mathematical calculations are used all the time to solve practical problems. However, calculations cannot answer empirical questions. Mathematical calculations provide the same sort of solutions to practical and mathematical problems. They provide a grammatical rule whose applications are propositions either inside or outside the calculus. This explains their capacity to help us say things about real objects and their properties.

The logicists' accounts of mathematical application, like those of Frege, Russell and Ramsey, treat calculations as universal propositions entailing predictions about the world. Wittgenstein challenges the logicists' view that the solution of a practical problem involves inferring a prediction about the world from mathematical calculations. For Wittgenstein, the question '*How* it is possible to infer a genuine proposition about the world from a mathematical calculation or statement?' is nonsensical. Mathematical propositions are not genuine propositions. They belong to a different logical space than propositions about genuine objects and events. Inference among mathematical propositions is calculus-bound. A mathematical proposition relates inferentially only to propositions within the same calculus. It does not entail or is entailed a proposition outside the calculus. It is impossible to infer a genuine proposition about something from a mathematical proposition about nothing. "The calculation is only a consideration of logical forms, of structures, and of itself can't yield anything new" [*Die Rechnung ist nur eine Betrachtung der logischen Formen, der Strukturen, und kann an sich nicht Neues liefern.* PG Pt. III §15 p. 604 (p. 307)]. The calculation (or associated mathematical statement) is not a premise in the solution of practical problems.

Wittgenstein does not erase the distinction between pure and applied mathematics. He challenges its interpretation. Logicists mistakenly view mathematics as a machine or tool made in preparation for some predefined use. "Here it is a matter of our concept of *application*. – We have an image of an engine which first runs idle, and then drives a machine." [*Hier handelt es sich um unsern Begriff der* Anwendung*. – Man hat etwa die Vorstellung von einem Motor, der erst leer geht, und dann eine Arbeitsmachine treibt.* PG §15 pp. 604, 606 (p. 309)] For Wittgenstein, mathematical calculations and grammatical rules share the same application. Their application is the construction and transformation of propositions. Since calculations themselves are rules of the calculus, they apply to themselves. Mathematical calculations are their own *internal* applications. The construction and transformation of propositions inside the calculus constitute its internal application. They also apply externally to genuine propositions or to propositions from other calculi. Because neither application is part of the calculus, application is not part of mathematics either. From the perspective of the calculus, pure and applied mathematics are not different.

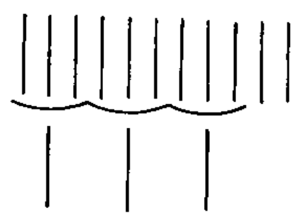
Application does not found the calculus. In solving practical problems, the calculation neither becomes empirical nor acquires some extra *reality* it lacked before. Attempts at formalizing the conditions of calculus application only result in extra calculi. However, the extra calculi do not provide a foundation for the original one. It does not make sense to make preparations for the application of arithmetics or any other mathematical calculus. The calculus is its own application. If the calculus exists, then at least one application of it exists: *itself*.

For this reason, Wittgenstein rejects the traditional view of consistency proofs. Using a calculus does not require a proof of its consistency. If consistency were necessary for the application of any calculus, it would not be a syntactic property. No calculation could prove it. On the other hand, if consistency meant only the absence of contradictions, consistency proofs would have no effect on the calculus. Thus, proving formally the applicability of a calculus is misguided and doomed to failure.

II. Calculations' Role in the Solution of *Practical* Problems

Hier kann man nun sagen: Die Arithmetik ist ihre eigene Anwendung. Der Kalkül ist seine eigene Anwendung. PG §15 p.608	At this point we can say: arithmetic is its own application. The calculus is its own application. PG §15 p. 310

Wittgenstein explains the role mathematical calculations play in the solution of practical problems, and the apparent difference between pure and applied mathematics in section III, 'The Foundations of Arithmetic' [*Die Begründung der Arithmetik*], of the *Philosophical Grammar*. In the relevant passages of this section, Wittgenstein considers set theory [§15 p. 606 (p.309)] and geometry [§17 pp. 626, 628 (pp. 319, 320)] in addition to arithmetic. However, his considerations of such cases are completely analogous to those of arithmetic. Since the rest of the dissertation has concentrated on arithmetic examples, this chapter focuses on the following passage from §15 of the *Philosophical Grammar*:



> Angenommen, mit dieser Rechnung wollte ich folgende Aufgabe lösen: Wenn ich 11 Äpfel habe und Leute mit je 3 Äpfeln beteilen will, wieviele Leute kann ich beteilen? Die Rechnung liefert mir die Lösung 3. Angenommen nun, ich vollzöge alle Handlugen des Beteilens und am Ende hätten 4 Personen je 3 Äpfel in der Hand. Würde ich nun sagen, die Ausrechnung hat ein falsches Resultat ergeben? Natürlich nicht. Und das heißt ja nur, daß die Ausrechnung kein Experiment har.
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> Es könnte scheinen, als berechtigte uns die mathematische Ausrechnung zu einer Vorhersagung, etwa, daß ich 3 personen werde beteilen können un 2 Äpfel übrigbleiben werden. So ist es aber nicht. Zu dieser Vorhersagung berechtigt uns eine physicalische Hypothese, die Außerhalb der Rechnung steht. Die Rechnung ist nur eine Betrachtung der logischen Formen, der Strukturen, und kann an sich nicht Neues liefern. [p. 602] [p. 602. Cf. Also PR §111 p. 122]

> Suppose I wish to use this calculation [*Rechnung*] to solve the following problem: if I have eleven apples and want to share them among some people in such a way that each is given three apples how many people can there be? The calculation supplies me with the answer 3. Now suppose I were to go through the whole process of sharing and at the end 4 people each had 3 apples in their hands. Would I then say that the calculation [*Ausrechnung*] gave a wrong result? Of course not. And that of course means that the calculation [*Ausrechnung*] was not an experiment.
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> It might look as though the mathematical calculation [*Ausrechnung*] entitled us to make a prediction, say, that I could give three people their share and there will be two apples left over. But that isn't so. What justifies us in making this prediction is an hypothesis of physics; which lies outside the calculation. The calculation is only a study of logical forms, of structures, and of itself can't yield anything new. [p. 307 Cf. also PR §111 pp. 132, 133]

The calculation's role in the solution of this problem intrigued Wittgenstein. Most of all, he was interested in the relationship between calculation and physical prediction. In this passage, Wittgenstein illustrated the philosophical differences between calculation and experiment by distinguishing numbers as *solutions* of practical problems and *results* of a calculation. In cases like Wittgenstein's example, the result of the calculation is also the solution to the problem. Number '3' is both the solution to the problem and the result of the calculation. However, Wittgenstein distinguishes these two roles of '3', while observing their close kinship.

A calculation's result may also be the solution of a non-mathematical problem, because it establishes the possibility of making non-mathematical predictions. This particular example predicts that to distribute twelve apples to at most *three* persons is possible. As such, the prediction may be true or false. For this reason, it makes sense to "suppose now, that I carry through the distribution and at the end there are four persons, each one with three apples in their hand." It is possible to imagine the prediction's falsity. It is possible to conjecture that the solution of a problem will fail. The calculation provides for this possibility. As such, the mathematical calculation cannot guarantee the prediction's success.

The calculation *alone* cannot produce anything new. The prediction's guarantee must lie outside the calculation.

Predictions are physical hypothesis outside the calculus. On the other hand, results belong to the calculus. A well-defined border separates these two logical spaces. The reason why Wittgenstein frames this distinction in the headline "The foundation of mathematics in which it is prepared for its applications" remains unexplained.

A. Wittgenstein's Criticism of The Logicists' Account of Mathematical Application

1. The Logicist's Puzzle

> In one sense there is no science of applied mathematics. When once the fixed conditions that any hypothetical group of entities are to satisfy have been precisely formulated, the deduction of the further propositions, which also will hold respecting them, can proceed in complete independence of the question as to whether or not any such group of entities can be found in the world of phenomena.
>
> A. N. Whitehead[1]

Logicists' explanation of mathematical application involves inferring physical predictions from mathematical calculations and propositions. The logicists would explain a case of mathematical application like that in *Philosophical Grammar* §15 through the inference of a physical prediction like 'If I have eleven apples and want to share them among some people in such a way that each is given three apples, there can be *three* people' from the mathematical equation 11 / 3 = 3. Justifying this inference challenges the logicists's view, because, for them, mathematical propositions like '11 / 3 = 3' are analytic, while propositions like 'If I have eleven apples . . .' are synthetic. Therefore, they must explain the possibility that an analytic mathematical proposition entails a synthetic one. In order to solve practical

1. Whitehead, Alfred North: "Foundations of Mathematics" article for the *Encyclopædia Britannica* (Chicago, 1988) http://www.britannica.com.

problems, mathematical propositions must "both be true regardless of fact and also imply a truth about . . . observable objects."[2] Alice Ambrose presents this puzzle as follows.

> How then can one account for the harmony between the two different areas of logic and empirical fact? How is it that we can apply arithmetical calculations to physical objects, or trigonometrical calculations to physical lines and angles? Is there a genuine mystery here or only a gratuitous puzzle?[3]

Two widespread myths about mathematics have stemmed from this puzzle. First, mathematics is a universal science. Mathematical propositions are universal. For example, equation '11 / 3 = 3' is a universal proposition about all possible additions. Second, there are propositions of so-called applied mathematics, which are neither genuine empirical propositions nor propositions of pure mathematics. Wittgenstein' account of mathematical application challenges these two myths.

2. First Myth: Mathematics is the Most Universal of Sciences

> Logic is concerned with the real world just as truly as zoology, though with its more asbtract and general features.
>
> Bertrand Russell[4]

Logicists mistakenly approach mathematics as the most abstract and universal of sciences. For Wittgenstein, mathematics is not about everything – as the logicists maintained, but about nothing. The universality of mathematics is no guarantee for its many applications, because mathematics is not more universal that its applications.

For Wittgenstein, mathematical operations are not universal. For example, mathematical addition is not an abstract generalization of all possible additions. Logicists believed

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2. Alice Ambrose, "Some Questions in Foundations of Mathematics" in Stuart Shanker: *Ludwig Wittgenstein. Critical Assessments. Volume Three: from the* Tractatus *to* Remarks on the Foundations of Mathematics: *Wittgenstein on the Philosophy of Mathematics*. (London: Croom Helm, 1986) 204. Originally published in *Journal of Philosophy* vol. 52 (1955) 197-213.

3. Ibid.

4. Bertrand Russell, *Introduction to Mathematical Philosophy* (1920) 169.

that whatever arithmetic says about addition applies to all possible additions. They thought that was the reason mathematics had such diverse applications, from social behavior to elementary particle physics. For them, mathematical propositions have an implicit hypothetical antecedent expressing their conditions of application. '4+4=8', for example, means that 'if '+' is an operation defined on the extension of concept Φ such that '+' obeys all the basic rules of arithmetical addition, according to the equivalence relation '=', then '4 Φs + 4 Φs = 8 Φs." According to the logicists, the arithmetical proposition '4+4=8' says that, for the proper sort of objects, adding four of them to another four results in four objects of that sort. For example, logicists believe that, since apples are among the proper sort of objects, '4 + 4 = 8' entails the *more particular* proposition that 'if one has four apples and adds them to other four apples, one will have eight apples'. For the logicists, '4 apples + 4 apples = 8 apples' is an application of '4 + 4 = 8'. The primary difference between the proposition about apples and the purely mathematical one is that the latter is a verifiable, physical hypothesis. An experiment can verify that adding four apples to four apples results in eight apples. In general, for the logicists, propositions in the arithmetics of natural numbers entail genuine propositions about apples. The application of the arithmetic of natural numbers to apples justifies this inference.

3. Wittgenstein against the Universality of Mathematics

For Wittgenstein, mathematics is not a universal science. Mathematical statements are not universal. The arithmetical proposition '4 + 4 = 8' is not about every addition. It is about arithmetical addition only. Furthermore, showing that apples are objects of the proper sort does not justify inferring '4 apples + 4 apples = 8 apples' from '4 + 4 = 8'. Adding the term 'apple' to a proposition in pure arithmetic does not create a new proposition of applied arithmetics about apples.

> Man muß sich aber davor hütten zu glauben “4 Äpfel + 4 Äpfel = 8 Äpfel¨” ist die konkrete Gleichung, dagegen 4 + 4 = 8 der abstrakte Satz, wovon die erste Gleichung nur eine spezielle Anwendung sei. So daß zwar die Arithmetik der Äpfel viel weniger algemein wäre, als die eigentliche allgemeine, aber eben in ihrem beschränkten Bereich (für Äpfel) gälte. – Es gibt aber keine “Arithmetik der Äpfel”, denn die Gleichung 4 Äpfel + 4 Äpfel = 8 Äpfel ist nicht ein Satz, der von Äpfeln handelt. [PG §15 p. 604]

> But we must be aware of thinking that “4 apples + 4 apples = 8 apples” is the concrete equation and 4 + 4 = 8 the abstract proposition of which the former is only a special case, so that the arithmetic of apples, though much less general than the truly general arithmetic, is valid in its own restricted domain (for apples). There isn’t any “arithmetic of apples”, because the equation 4 apples + 4 apples = 8 apples is not a proposition about apples. [PG §15 p. 308]

Wittgenstein claims that “there isn’t any *arithmetic of apples,*” meaning that no third realm of applied mathematics exists between mathematics and the real world. It does not make sense to talk about propositions like ‘4 apples + 4 apples = 9 apples’ as being neither mathematical nor genuine. No middle ground between genuine and mathematical propositions exists.[5]

4. Letters and Schemes.

Dispelling the myth that mathematical propositions are universal requires reinterpreting the role of quantification and variables in mathematics. Wittgenstein does this in sections XIII and XIV, Part II of *Philosophical Grammar*, by focusing on the syntactic role of letters. In paragraph 150, Wittgenstein described the three possible functions of a letter in mathematics: (1) as general constant, (2) as unknown and (3) as marker for a blank space. In the first case, the letter belongs to the language of the calculus. In the other two cases, it is an external element. In either case, every letter in a calculation statement is a general constant. For Wittgenstein, there is no significant difference between general constant and

5. For Wittgenstein, ‘4 apples + 4 apples = 8 apples’ is not a genuine proposition about apples. It is the mathematical proposition 4 + 4 = 8 expressed in terms of apples instead of numbers.

'universal' variables. All mathematical variables in a calculation statement are universally quantified, not only those under the explicit scope of a quantifier. However, in mathematics, 'universal quantification' does not mean universality in the platonist sense. Wittgenstein also rejected the traditional interpretation of the universal quantifier in mathematics. In note I to paragraph 150 of the *Philosophical Remarks*, he wrote:

> Dieses zeichen '(x)' sagt aber gerade das Gegentiel dessem, was es in den nicht mathematischen Fällen sagt . . . nämlich gerade, daß wir die Variable in dem Satz als *Konstante* auffassen sollen. [PR §150 n. I, p. 164]

> But this sign '(x)' says exactly the opposite of what it says in non-mathematical cases . . . i.e. precisely that we should treat the variables in the proposition as *constants.* [PR §150 n. I, p. 174]

For Wittgenstein, letters in mathematics and logical notation have radically different meanings. Disregarding this difference produces the mistaken idea that variables may occur unbound. Logical formulae represent the logical form of genuine propositions. Since genuine propositions can be general, the ability to express generality in the logical formalism is necessary. This is the role of letters in logical formalism. However, mathematical formulas do not represent the logical form of genuine propositions. Accordingly, their letters have a different role. Talk about *all* the numbers, for example, may suggest that mathematical propositions are universal, but mathematical generality is of a different sort. Mathematical generality joins totality and necessity in a single notion. "For in mathematics '*necessary*' and '*all*' go together. (Unless we replace these idioms throughout by ones which are less misleading.)"[6] Mathematical modality is not that of possible / necessary or universal / particular, but sense / nonsense. Mathematical modality is *syntactic*.

According to the previous analysis, a mathematical formula has three different interpretations depending on the syntactic role of its letters. This classification corresponds

6. PR §150

to a division in the possible meaningful questions about such expressions:[7]

(1) If all its letters are constants in the calculus, the formula is a calculation proposition. It makes sense to ask whether the formula is correct or not.[8] In the case of equations, the answer to this question depends on whether or not the rules of the calculus allow for each side of the '=' sign to transform into the same expression.[9]

(2) If at least one of the letters expresses an unknown and no letter marks a blank space, then the formula is not a proposition but a scheme. Asking if the equation is solvable makes sense. The equation is solvable only if the replacement of the unknown letter for a general constant (not necessarily a letter) results in a true proposition.

(3) If at least one of the letter marks a blank space, then the expression is incomplete. It makes sense to ask if it is syntactically permissible. An incomplete expression is syntactically permissible if it is possible to construct a well-formed formula by filling its blank spaces.

Wittgenstein adopts this classification to dispel the myth that mathematics has different levels of generality. Expressions with letters are no more general than expressions with other mathematical constants, like numerals. In a footnote to paragraph 150 of the *Philosophical Remarks*, Wittgenstein notes,

> Ich habe noch zu wenig betont daß 25 x 25 = 625 auf *genau* derselben Stufe und von *genau derselben Art* ist wie $x^2 + y^2 + 2xy = (x+y)^2$. [PR §150 n.1, p. 164]

> I still haven't stressed sufficiently that 25 x 25 = 625 is on *precisely* the

7. In strict sense, it is not any mathematical *equation*, but any expression *in the form of* an equation, since it may turn out to be that the given expression is not a genuine equation.

8. Unless, of course, the equation contains other letters, which are not constant variables.

9. PR 154

same level as and of *precisely the same kind* as $x^2 + y^2 + 2xy = (x+y)^2$. [PR §150 n. 1, p. 174]

Wittgenstein illustrates this distinction with the expression '$a+(b+c)=(a+b)+c$'. This formula expresses both an algebraic proposition and the law of associativity for arithmetical addition. As an algebraic proposition, its letters are general constants. In the law of associativity, they are schematic letters, and the expression is a scheme.

The law of additive associativity in arithmetic '(a+b) + c = a + (b+c)' seems to express a general property of all numbers or all additions. However, it only appears to be general. Distinguishing cases or instances from applications is crucial to understanding the apparent generality of mathematical propositions. Consider the following four propositions:

(1) (a+b) + c = a + (b+c)

(2) (3+4) + 6 = 3 + (4+6)

(3) All camels are herbivores.

(4) My camel is a herbivore.

Mathematical propositions as grammatical rules are *applied*. Genuine generalizations have instances. (2) is an application of (1), while (4) is an instance of (3). The truth of genuine generalizations like (3) rests on induction from particular cases like (4). The truth of grammatical claims like (1) does not depend in anyway on that of propositions like (2). (2) in no way confirms (1). Even if proposition (2) occurs in the proof of (1), it would not be a confirming instance. That is why mathematical induction is so different from genuine inductions as they feature in empirical science. In mathematics, instances do not exist. Hence, they cannot confirm any generalizations. Generality holds throughout all mathematical propositions. Proposition (2) is not less general than (1), even though it may seem that (2) is about particular numbers and (1) is about all of them.

In mathematics, generalization from particular cases is impossible. From a series of known mathematical propositions, to come up with a more general proposition by generalizing into similar cases is impossible. Talk of 'similar cases' involves classifying the propositions through a general concept. However, mathematical concepts are disjunctions of their members. Bringing mathematical propositions together under a mathematical concept would result in a conjunction of the original propositions. A conjunction is not more general than the sum of its elements, though. In consequence, bringing mathematical propositions together under a mathematical concept does not result in a more general mathematical proposition. The resulting mathematical statement is neither less nor more necessary or general that its elements.

5. Wittgenstein and Ramsey on *Anwendung* and Interpretation

Wittgenstein entitled §15 of section II of his *Philosophical Grammar* '*Die Begründung der Arithmetik, in der diese auf ihre Anwendungen vorbereitet wird (Russell, Ramsey)*' Anthony Kenny translated it for the English edition of *Philosophical Grammar* as 'Justifying arithmetic and preparing it for its application (Russell, Ramsey)'. However, a better translation would be 'The founding of arithmetic in which it [arithmetic] is prepared for its applications'. Kenny's translation misleads in important ways. Kenny's title suggests that Wittgenstein intends to take up the separate but related topics of justifying arithmetic and of "preparing arithmetic for its application" whatever that would be. The alternative translation reveals Wittgenstein's specific target: Russell and Ramsey's approach to the foundations of arithmetic, according to which a fully satisfactory foundation for arithmetic requires and includes provision to use logic to specify and control the application of arithmetic.

Evidence suggests that Wittgenstein directed his comments on this topic at Ramsey's "The Foundations of Mathematics" (1925). First of all, 'The Foundations of Mathematics' is the section's title. Second, Ramsey presented in "The Foundations . . ."

the identity theory Wittgenstein discusses in §16. Third, most of the quotations and examples Wittgenstein uses in this section come from Ramsey's article. For example, the use of apples as an example of the application of arithmetics originally occurs in Ramsey.

Mathematical application is not one of "The Foundations" central concerns. On a superficial reading, Ramsey's text does not say anything about the application of mathematics. The very first sentence of his introduction states that it pertains to "the general nature of pure mathematics" instead of *applied* mathematics. Nevertheless, in the very next paragraph, he states that any theory of mathematical concepts is "hopeless" if it only accounts for the meaning of mathematical terms in mathematical propositions.

> . . . for these occur not only in mathematical propositions, but also in those of everyday life. Thus '2' occurs not merely in '2 + 2 = 4', but also in 'It is 2 miles to the station'. . . Nor can there be any doubt that '2' is used in the same sense in the two cases, for we can use '2 + 2 = 4' to infer from 'It is two miles to the Gogs via the station', so that these ordinary meanings of two and four are clearly involved in '2 + 2 = 4'.[10]

Ramsey's interest in his *Foundations of Mathematics* is the interpretation of mathematical formulae. In this article, perhaps for the first time in the history of logicism, Ramsey laid out a separation between formalism and interpretation. Ramsey distinguished between the formal elements of a theory – formal language, axioms and rules, and its interpretation – a mathematical domain with special, designated functions and relations defined over it. Ramsey makes it clear that systematizing the formalism alone, as Russell and Whitehead did in *Principia Mathematica*, is not sufficient for mathematics' foundation. For Ramsey, the foundations of mathematics require guaranteeing the proper interpretation, too. Described in contemporary terms, the proper interpretation of a formalism must have the property of satisfying the axioms of the formalism. Ramsey's concern for the interpretation of mathematical calculi focused on formalizing their application conditions. For Ramsey, the

10. Frank Ramsey, "The Foundations of Mathematics" p. 165

application of the formalism to the objects, functions, relations and propositions in its specification requires a proper interpretation of the formalism.[11]

From Ramsey's standpoint, the application of natural number arithmetic requires the construction of a proper interpretation of every arithmetic function and operation. For Ramsey, applying natural number arithmetic to the distribution of apples, for example, requires a specific interpretation of the equivalence relation of *numerical equality* among groups of apples.[12] It is also necessary to specify interpretations for the mathematical operations of the arithmetic calculus. The operation of putting extra apples in a group of apples may interpret the arithmetical operation of addition, for example. The operation of taking apples from a group of apples may interpret the operation of substraction, and so forth. Ramsey thought that, for the proper interpretation, these operations must follow the basic rules of the arithmetic operation. In the case of addition, for example, the interpretation must be associative, commutative, have a neutral element, et cetera.

For Wittgenstein, Ramsey's formal interpretation is a superfluous and misguided attempt at "preparing a calculus for its application." First of all, it is doubtful whether or not a full correspondence between pure, natural number arithmetic and some "arithmetic of apples" is possible. Furthermore, such correspondence would eliminate the putative difference between *pure* and *applied* arithmetic. If the interpretation of arithmetical operations on apples corresponded totally to the operations on natural numbers, both calculi would be as

11. Wittgenstein is not confusing the notions of interpretation and application. On the contrary, his criticism of Ramsey's position requires a clear separation between them. Wittgenstein criticizes Ramsey's idea that the provision of a proper formal interpretation of the calculus prepares it for its application.

12. Another important aspect of Wittgenstein's philosophy of arithmetic is its separation from *counting*. In Part II, section 21 of the *Philosophical Grammar*, Wittgenstein argues that, in arithmetic, counting is not a more primitive notion that numerical equality.One can tell that two groups are the same in number, without actually knowing which number this is. Wittgenstein takes it a step further to say that 'numerical equality' is not about numbers at al, even if its surface grammar suggests so. Talking about two groups having the same quantity of objects, misleads one into thinking that such thing as their common quantity exists.

general. It would not make sense to say that arithmetics with apples is a particular case of the more general arithmetics with numbers. Applied calculi are not less general than non-interpreted ones. Mathematics has no hierarchies of generality. They are all at the same level. If groups of apples followed the same arithmetical rules as natural numbers, they would *be* natural numbers themselves. “Calculation with apples is essentially the same as calculation with lines or numbers” [*Die Rechnung mit Äpfeln ist wesentlich dieselbe, wie die mit Strichen oder Ziffern.* PHG §15 p. 608 (p. 310)]. On the other hand, if elementary arithmetic and its interpretation on apples did not follow the same rules, they would be two different calculi. In either case, neither calculus could justify the other.

B. Wittgenstein’s Account of Mathematical Application

The main difference between Wittgenstein’s and the logicists’ account starts with the very interpretation of the non-mathematical problem. Taking a closer look at the relevant passage from §15 of the *Philosophical Grammar,* (or §111 of the *Philosophical Remarks)*, three elements are distinguishable:

1. The problem [*Aufgabe*]: “If I have eleven apples and want to share them among some people in such a way that each is given three apples how many people can there be?” This is a general hypothetical question. Furthermore, it is a modal question asking what is *possible.* It asks for the number of people it is possible to give three apples from a group of eleven apples.

2. The solution [*Lösung*]: Wittgenstein makes it clear that the solution to the problem is the number 3, not that there can be 3 persons.

3. The prediction [*Vorhersagung*] that I could give three people their share of four apples, leaving two apples.

Wittgenstein’s view of mathematics as grammar addresses the relation between the calculation and these three different elements. In the aforementioned passage, he explicitly says that the calculation *supplies* [*liefert*] the solution to the problem. It does not *entitle* [*berechtigen*] one in making the prediction. Calculation does not justify the solution of practical problems. Providing the solution to a non-mathematical problem is essentially different than justifying a non-mathematical prediction. Wittgenstein agrees that the calculation in his example says that 11 ÷ 3 = 3 – calculation and equation are identical, but it does not predict that if one gave three people their share, “there will be two apples left over.” Still, it says that it is possible to share eleven apples among three people in such a way that each receives three apples. However, the latter is not a prediction, but a grammatical proposition.

The calculation says that it is possible to share eleven apples among three people in such a way that each receives three apples. The question ‘If I have eleven apples and want to share them among some people in such a way that each receives three apples how many people can there be?’ is a grammatical question. It asks what the largest possible number of people is that could receive three of eleven apples. This possibility is not physical. It is grammatical.“For the word “can” in that proposition doesn’t indicate a physical (physiological, psychological) possibility.” [*Denn das wort “kann” in diesem Satz deutet nicht auf eine physiche (physiologische, psychologishe) Möglichkeit.* PG §14 p. 596 (p. 304)]

The proper answer to the question “if I have eleven apples and want to share them among some people in such a way that each receives three apples, how many people can there be?” is not a universal statement about apples and their distribution. It is grammatical, that is, mathematical. Its solution must be an appropriate grammatical rule. The mathematical calculation provides this rule. The calculation gives the proper solution, because the question is grammatical. It is not a question about the necessary properties of apples or their distribution, but a grammatical question of what makes sense to predict. The answer is a rule for the use of the word ‘apple’. It says that the genuine proposition “I give three people their share and there will be two apples left over” is grammatically correct. The prediction “that I could give three people their share and there will be two apples left over” makes sense.

For Wittgenstein, the prediction about apples and their distribution is not the solution to the problem, but its *application.* The prediction that “I could give three people their share and there will be two apples left over” is not the solution of the practical problem. The solution to the mathematical problem precedes the physical prediction. The formulation of the prediction requires the mathematical calculation, because it is its application. The calculation provides the grammar of the genuine proposition. Formulating the prediction involves applying the calculation as a grammatical rule. Thanks to the calculation, the prediction that three people will receive their share of four apples with two apples left makes sense. Only in this sense do mathematical calculations apply to the solutions of non-mathematical problems. To apply a mathematical calculation means using it as a grammatical rule in the construction or transformation of propositions. In §107 of the *Philosophical Remarks*, Wittgenstein writes,

> Die arithmetischen Sätze dienen, wie Multiplikationstabellen und dergleichen, oder auch wie Definitionen, auf deren beiden Seiten nicht ganze Sätze stehen, zur *Anwendung* auf die Sätze. Und auf etwas anderes kann ich sie ja sowieso nicht anwenden. (Ich brauche also nicht erst irgendwelche Beschreibung ihrer Anwendung.) [PR §107 p. 119]

> Arithmetical propositions, like the multiplication table and things of that kind, or again like definitions which do not have whole propositions standing on both sides, are used in *application* to propositions. And anyhow I certainly can’t apply them to anything else. (Therefore I don’t first need some description of their application.) [PR §107 p. 129]

The application of a mathematical calculus to external propositions requires embedding the rules of one calculus into the grammar of the other. In particular, a calculus application to genuine propositions requires embedding the calculus rules into the grammar of natural language. Thus, calculation rules become grammatical rules of natural language. The calculation $11 \div 3 = 3$ is not only a rule in the arithmetic of natural numbers, but also a grammatical rule of English. The construction of the English sentence, “I could give three people their share and there will be two apples left over” is one of its applications. According to Friedrich Weissmann’s notes, during a conversation at Schlick’s home on December 28, 1930, Wittgenstein said,

> Was heißt es, einen Kalkül anwenden? . . . Man wendet den Kalkül in der Weise an, daß er die Grammatik einer Sprache ergibt. Dem, was die Regel erlaubt oder verbietet, entspricht dann in der Grammatik das Wort ‘sinvoll’ und ‘sinloss’. [PR Appendix II, section ‘Widerspruchsfreiheit’ p.309]

> What does it mean to apply a calculus? . . . We apply the calculus in such a way as to provide the grammar of a language. For, what is permitted, or forbidden by the rukes then corresponds in the grammar to the words ‘sense’ and ‘senseless’. [PR Appendix II, section ‘Consistency’ p.322]

In Wittgenstein’s example, the calculation provides the grammar of the genuine proposition about apples. The proposition makes the physical prediction. The calculation makes the prediction possible, but the prediction’s truth remains independent of the mathematical calculation. It requires further empirical testing. The calculation itself predicts nothing about apples.

In §17 of the *Philosophical Grammar*, Wittgenstein writes,

> (Ein Satz, der auf einer falschen Rechnung beruht (wie etwa “er teilte das 3 m lange Brett in 4 Teile zu je 1 m”) ist unsinnig und das beleuchtet, was es heißt “Sinn haben” und “etwas mit dem Satz meinen”) [PG §17, p. 626]

> (A statement based on a wrong calculation (such as “he cut a 3-metre board into 4 one metre parts”) is nonsensical, and that throws light on what is meant by “making sense” and “meaning something by a proposition”).

Applying a correct calculation results in a well-formed statement. Applying an incorrect calculation results in a nonsensical one. If the calculation is correct, as in the example on

§15, the proposition makes sense. If the calculation is incorrect, as in the example from §17, the proposition is nonsensical. In either case, the calculation does not make the genuine proposition true or false. The mathematical calculation has nothing to do with the truth of its application. It does not justify it. A correct calculation may yield a false non-mathematical proposition, as well as a true one. The truth of the non-mathematical proposition resulting from the application of the calculation is independent of the calculation.

III. *Anwendung* and the Foundations of Mathematics

Unter Anwendung verstehe ich das, was die Laut-verbindungen oder Striche überhaupt zu eine Sprache macht. In dem Sinn, in dem es die Anwendung ist, die den Stab mit Strichen zu einem *Maßstab* macht. Das *Anlegen* der Sprache an die Wirklichkeit. PR §54 p. 74	By application I understand what makes the combination of sounds or marks into a language at all. In the sense that it is the application which makes the rod with marks on it into a *measuring rod*: *putting* language *up against* reality. PR §54 p. 84

From the perspective of the calculus, pure and applied mathematics are not significantly different. Mathematical problems solve mathematical and non-mathematical problems in the same way. However, they are critically different from the perspective of *Anwendung.* In the solution of a problem in pure mathematics, the application of the calculation happens inside the calculus. The calculation is its own application. The solution of a non-mathematical problem applies the calculation to a genuine proposition outside the calculus.

A. The Autonomy of Mathematical Calculi

> Der Kalkül setzt den Kalkül voraus. [PR §108 p. 120]
>
> The calculus presupposes the calculus. [PR §109 p. 130]
>
> Jede Rechnung der Mathematik ist eine Anwendung ihrer selbst und hat nur als solche Sinn. [PR §109 p. 120]

> Every mathematical calculation is an application of itself and only as such does it have a sense. [PR §109 p. 130]

> Hier kann man nun sagen: Die Arithmetik ist ihre eigene Anwendung. Der Kalkül ist seine eigene Anwendung. [PG §15 p. 608]

> At this point we can say: arithmetic is its own application. The calculus is its own application. [PG §15 p. 310]

In section III of the *Philosophical Grammar*, Wittgenstein states that mathematical calculi are their own applications. Since mathematical calculations are also the rules of the calculus they belong to, they apply to themselves. Mathematical propositions are grammatical rules that govern the same language that expresses them. This latter sort of grammatical rule is common. For example, the statement ‘In English, the first word of every sentence is capitalized’ expresses a grammatical rule in the grammar of English. The rule applies to sentences in that language. In particular, it applies to the sentence that expresses it. However, neither the English sentence nor the grammatical rule apply to themselves. For that, the sentence would have to be autonomous [*autonom*], like an arithmetic calculation or a geometrical construction.

> Der Sinn der Bemerkung, daß die Arithmetik eine Art Geometrie sei, ist eben, daß die arithmetischen Konstruktionen autonom sind, wie die geometrischen, und daher sozusagen ihre Anwendbarkeit selbst garantieren.
>
> Denn auch von der Geometrie muß man sagen können, sie sei ihre eigene Anwendung. [PG §15 pp. 600, 602 Cf. PR §111 p. 112]

> The point of the remark that arithmetic is a kind of geometry is simply that arithmetical constructions are autonomous like geometrical ones and hence, so to speak, themselves guarantee their applicability.
>
> For it must be possible to say of geometry too that it is its own application. [PG §15 pp. 306, 307 Cf. PR §111 p.132]

This difference between grammatical statements like ‘In English, the first word of every sentence is capitalized’ and mathematical calculations is critical. Mathematical calculations are autonomous, while sentences that express grammatical rules of their own language are not. If the aforementioned sentence did not follow the rules of English, it would not make sense. However, the grammatical correctness of the statement does not guarantee that the rule expressed is an actual rule of English. By contrast, calculations cannot be grammatically correct unless they are rules of their calculus. The aforementioned English sentence and its negation are both grammatically correct. However, only one of them expresses a grammatical rule of English. By contrast, every correct mathematical calculation is a rule of the calculus. For a calculation, being correct, obeying the rules of the calculus and being a rule are the same.

Calculations are autonomous, because they do not express mathematical rules. They are the rules themselves. Grammatical rules are different from the English sentences that express them. Grammatical rules of language are not autonomous, while calculations are.

> Die arithmetik hat es mit dem Schema | | | | zu tun. – Aber redet denn die Arithmetik von Strichen, die ich mit Bleistift auf papier mache? – Die Arithmetik redet nicht von den Strichen, sie *operiert* mit ihnen. [PG §19 p. 654]

> What arithmetic is concerned with is the schema | | | |. – But does arithmetic talk about the lines I draw with pencil on paper? – Arithmetic doesn't talk about the lines, it *operates* with them. [PG §19 p. 333]

The arithmetical calculation in Wittgenstein's example of §15 (see above display) is autonomous, because it is not about strokes or their division. It is a division itself. Dividing the strokes into groups of three is performing the calculation. The calculation is not *about* the division. The calculation *is* the division. It involves applying the rules for writing strikes and dividing them. The calculation is both one of the arithmetical rules for division and an application of them. Divisions are rules of division. Arithmetical calculations are rules of arithmetic. Calculations are rules of the calculus. Such is the autonomy of arithmetic.

B. *Anwendung* is an Essential Feature of Mathematics

Wenn man sagt: “es muß der Mathematik wesentlich sein, daß sie angewandt werden kann”, so meint man, daß diese Anwend*barkeit* nicht die eines Stückes Holz ist, von dem ich sage “das werde ich zu dem und dem anwenden können.”

PG §17 p. 626

If we say “it must be essential to mathematics that it can be applied” we mean that its applic*ability* isn’t the kind of thing I mean of a piece of wood when I say “I will be able to apply it to this and that.”

PG §17 p. 319

Logicists confuse two different kinds of grammatical application: external and internal. The external application gives rules for embedding the grammar of one language into that of another, quite different language. The internal application lives within the language. In either case, grammatical rules apply to propositions. As the name suggests, mathematical rules apply internally to propositions inside the calculus, and externally to propositions outside the calculus. However, external application is not part of the calculus itself. It lies entirely outside the calculus. Neither sort of application is mathematical. The only mathematical part of applied mathematics is the calculus.

> Die Grammatik is für uns ein reiner Kalkül. (Nicht die Anwendung eines auf die Realität.) [PG §15 p. 612]
>
> Grammar is for us pure calculus (not the application of calculus to reality). [PG §15 p. 312]

From the perspective of the calculus, no significant difference exists between both applications. The rules of the calculus apply equally to propositions inside and outside the calculus. However, the external application connects the calculus with reality. The external application allows the use of calculations to solve non-mathematical problems. External application is an essential element of mathematics. It distinguishes mathematics from games.

The rest of mathematics is pure calculation [*Berechnung*]. In *Philosophical Grammar* §11,[13] Wittgenstein writes the following.

> Wenn ich in unserem Spiel 21 x 8 ausrechne, und wenn ich es tue, um damit eine praktische Aufgabe zu lösen, so ist jedenfalls die Handlung der Rechnung in beiden Fällen die Gleiche (und auch für Ungleichungen könnte in einem Spiele Platz geschaffen werden.) Dagegen ist mein übriges Verhalten zu der Rechnung jedenfalls in den zwei Fällen verschieden.
>
> Die Frage ist nun: kann man von dem Menschen, der im Spiel die Stellung "21 x 8 = 168" erhalten hat, sagen, er habe herausgefunden, daß 21 x 18 = 168 sei? Und was fehlt ihm dazu? Ich glaube, es fehlt nichts, es sei denn eine Anwendung der Rechnung.
>
> Die Arithmetik ein Spiel zu nennen, ist ebenso falsch, wie das Schieben von Schachfiguren (den Schachregel gemäß) ein Spiel zu nennen; denn das kann auch eine Rechnung sein. [PG §11 p. 573]

> When I work out 21 x 8 in our game the steps in the calculation, at least, are the same as when I do it in order to solve a practical problem (and we could make room in a game for inequations also). But my attitude to the sum in other respects differs in the two cases.
>
> Now the question is: can we say of someone playing the game who reaches the position "21 x 8 = 168" that he has found that 21 x 8 = 168? What does he lack? I think the only thing missing is an application for the sum.
>
> Calling arithmetic a game is no more or less wrong than calling moving chessmen according to chess rules a game; for that might be a calculation too. [PG §11 p. 292]

Using calculations to solve mathematical, as well as non-mathematical problems is essential to mathematics. A calculus without external application would not be mathematical. External application provides the calculus with significance [*Bedeutung*]. It gives it certain 'importance for life' [*Lebenswichtigkeit*].

> Es ist den Leuten unmöglich, die Wichtigkeit einer tatsache, ihre Konsequenzen, ihre Anwendung, von ihr selbst zu unterschieden; die Beschreibung einer Sache von der Beschreibung ihrer Wichtigkeit. [PG §11 p. 578]
>
> People cannot separate the importance, the consequences, the application of a

[13]. The section's title is 'The comparison between mathematics and a game' [*Die Mathematik mit einem Spiel Vergleichen*].

> fact from the fact itself; they can't separate the description of a fact from the description of its importance. [PG §11 p. 295]

While Wittgenstein finds the external application of the calculus essential to mathematics, he does not find it foundational. Application does not play a foundational role in the calculus. In the solution of non-mathematical problems, the calculation neither becomes empirical nor acquires some extra *reality* it lacked before. The external application of a calculus does not make it more *real*.

> Die unrichtige Idee ist, daß die Anwendung eines Kalküls in der Grammatik der wirklichen Sprache, ihm eine Realität zuordnet, eine Wirklichkeit gibt, die er früher nicht hatte. [PG §15 p. 610]

> What is incorrect is the idea that the application of a calculus in the grammar of real language correlates it to a reality or gives it a reality that it did not have before. [PG §15 p. 311]

Making preparations for the application of arithmetic or any other mathematical calculus does not make sense. Since arithmetic is its own internal application, if the calculus exists, it has at least one application in itself. Application takes care of itself. For Wittgenstein, the logicists' project of circumscribing the totality of possible external applications of arithmetic using mathematical tools is impossible. The only way of picking out all of the legitimate applications of arithmetics would be using the expression 'legitimate application of arithmetic'.

> Man könnte sagen: Wozu die Anwendung der Arithmetik einschränken, sie sorgt für sich selbst. (Ich kann ein Messer herstellen ohne Rücksicht darauf, welche Klasse von Stoffen ich damit werde schneiden lassen; das wird sich dann schon zeigen.) [PG §15 p. 601]

> You could say: why bother to limit the application of arithmetic, that takes care of itself. (I can make a knife without bothering about what kinds of materials I will have cut with it; that will show soon enough.) [PG §15 p. 306]

Just like making a knife without considering what kind of materials it will cut, it is possible to construct a calculus without considering what kind of non-mathematical problems it will solve. The application is entirely external to the calculus. The calculus is independent of its application. A calculus without external application is no less a calculus than an externally applied one.

C. On Consistency [*Widerspruchsfreiheit*]

Ich habe eine Arbeit von Hilbert gelesen über die Wiederspruchfreiheit. Mir kommt vor, daß diese ganze Frage falsch gestellt ist. Ich möchte fragen: *Kann* denn die Mathematik überhaupt widerspruchsvoll sein? Ich möchte die Leute fragen: Ja, was tut ihr denn eigentlich?	I've been reading a work by Hilbert on consistency. It strikes me that this whole question has been put wrongly. I should like to ask: *Can* mathematics be inconsistent at all? I should like to ask these people: Look, what are you really up to?
PR App. II. p. 305	PR App. II. p. 318.

The German word for consistency is '*Widerspruchsfreiheit*', which literally means 'freedom of contradiction'. For Wittgenstein, the notion of '*Widerspruchsfreiheit*' confuses two independent notions: the quality of a calculus being free from contradictions, and the necessary conditions for applying a calculus. For Wittgenstein, the two clearly do not match. The applicability of a calculus does not require the absence of contradictions. Mathematicians' use of '*Widerspruchsfreiheit*' misleads by implying that a calculus needs to be free from contradiction in order to be applicable. Furthermore, neither of the two notions confused in the foundational role of *Widerspruchsfreiheit* is provable. Hilbert and Ramsey's demand for consistency proofs as part of the foundation of mathematics is twice mistaken. On the one hand, application requires no proof of consistency. Since calculation is its own application, the applicability of the calculus needs no further proof but its own existence.

> Wie wuare es denn, wenn ich einen solchen Kalkül anwenden will? Hätte ich bei der Anwendung kein gutes Gewissen, solange ich nicht die Widerspruchsfreheit bewiesen habe? Aber kann ich denn so fragen? Kann ich den Kalkül anwenden, so habe ich ihn eben angewendet; es gibt kein nachträgliches Korrigieren. Was ich kann, das kann ich. Ich kann die

> Anwendung nicht dadurch ungeschehen machen, daß ich sage: eigentlich war das keine Anwendung. [PR Appendix II p. 319]

> But suppose I want to apply such a calculus? Would I apply it with an uneasy conscience if I hadn't already proved [its consistency]? but how can I ask such a question? If I can apply a calculus, I have simply applied it; there's no subsequent correction. What I can do, I can do. I can't undo the application by saying: strictly speaking that wasn't an application.. [PR Appendix II p. 332]

Formal proof cannot demonstrate the applicability of a calculus. Applicability is not a syntactic feature of the calculus. As a calculation, no proof can establish anything about its calculus or formal system. It cannot determine if the calculus is applicable or not. Any attempts at proving the applicability of a calculus formally will fail.

On the other hand, if 'consistency' consisted of the absence of contradictions, it would not be provable either. First of all, it is impossible to formulate contradictory rules. Formulating a rule is performing it as calculation, which requires its application as rule. To formulate a calculation rule, it must be applicable. In consequence, contradictory formulas do not exist. No rule can contradict itself or another rule.

> Warum dürfen sich Regeln nicht widersprechen? Weil es sonst keine Regeln wären. [PG §14 p. 598]

> Why may not the rules contradict one another? Because otherwise they would not be rules. [PG §14 p. 305]

For Wittgenstein, if inconsistency is the existence of a proposition like 'p · ~p' or '2 x 2 = 5' among the rules of the calculus, it is not a 'great misfortune' [*großes Unglück*]. The existence of such a rule cannot 'harm' [*schaden*] the calculus. It cannot make it useless or inapplicable. The existence of the rule sufficiently guarantees its applicability.

> Wie wäre es etwa, wenn man in der Arithmetik zu den üblichen Axiomen die gelichung 2 x 2 = 5 hinzunemmen wollte? Das heiße natürlich, daß das Gleichzeichen nun seine Bedeutung gewechselt hätte, d. h. Daß nun andere Regeln für das Gleichzeiten gälten. [PG §14 p. 595]

> Suppose someone wanted to add the usual axioms of arithmetic the equation 2 x 2 = 5. Of course that would mean that the sign of equality had changed

> its meaning, I. e. That there would now be different rules for the equal sign.
> [PG §14 p. 303]

Using calculation to decide on a philosophical problem is the common mistake made in attempts to found mathematics on consistency proofs. In this section of the *Philosophical Grammar*, Wittgenstein clarifies a confusion in the philosophy of mathematics by separating philosophy from mathematics, and “putting each one in its place.” Wittgenstein does not deny the formal results of his adversaries. He challenges their philosophical *prose*. In this section in particular, he reclaims the notion of *Anwendung* for philosophy. He separates the philosophical problem of application from the formal concerns of consistency, interpretation, etc. Furthermore, he fully divorces the calculus itself from its application. This separation allows him to explain the joint autonomy and applicability of mathematical calculations. It explains how mathematical calculations are about nothing and still solve practical problems. It also explains how mathematical calculations can be both rules of mathematical calculus and syntactical rules of natural language.

IV. Conclusion

Wittgenstein bases his philosophy of mathematics during the thirties on the strong importance of context. For Wittgenstein, understanding the meaning of a proposition requires an analysis of its role in a larger system of propositions or in other sentences. Accordingly, the philosophy of mathematics must start by analyzing the contexts in which mathematical propositions are used.[14] For Wittgenstein, as well as Ramsey, a philosophical analysis of numbers must contain an understanding of both their occurrences in purely mathematical contexts and in non-mathematical ones. In particular, understanding the meaning of

14. The importance of propositions’ context of use will become more evident in Wittgenstein’s later work. However, Wittgenstein’s appreciation of the importance of mathematical calculi already shows recognition of the importance of context for understanding propositions.

numerical statements [*Zahlangaben*] requires an analysis of their roles in the contexts of their use. In the case of mathematical *Zahlangaben*, it requires understanding their role in calculation. In the case of non-mathematical ones, it requires an analysis of their role in the application of mathematics. Both analyses are essential for the full understanding of numbers and mathematics.

This chapter is an analysis of mathematical propositions and calculations in the context of their application. The grammatical nature of mathematical propositions manifests itself in their application. In Pt. II, section III of *Philosophical Grammar*, Wittgenstein offers an account of mathematical application where mathematical propositions are rules of grammar. They provide the rules for the creation and transformation of sentences, either in the calculus (*internal* application) or outside it (*extrenal* application).

The *Anwendung* process starts with calculation. If a calculation is performed in accordance with the rules of the calculus, it is correct. Every calculation leaves behind a trace. If the calculation is correct, its trace is a (true) mathematical proposition. That proposition is its internal *Anwendung*. If the calculation is correct, the proposition is grammatically correct and, in consequence, a rule of its calculus. Applying this calculation externally requires embedding the calculus rules in another grammar, like natural English grammar. Mathematical propositions provide the grammar of English sentences. If the calculation is correct, the sentences are grammatically correct and express genuine propositions. The calculation guarantees that what they say is possible, but it does not justify them or guarantee their truth. If the calculation is incorrect, the expressions are nonsense. Next chapter gives a formal and detailed account of how this process takes place.