Chapter 6

Mathematics as Grammar

A formal treatment

I. Introduction

The question, ‘Is mathematics part of the formal grammar of language?’ has been part of the philosophical debate on mathematics for more than a century. It lies at the center of the debate between Carnap and Bar Hillel, on the one side, and Gödel, Tarski and Quine on the other. The previous two chapters provided the background for an answer to those questions. The third chapter explained the relationship between mathematics and natural language, while the fourth one provided a formal model of grammatical analysis. An accurate picture of the relationship between mathematical calculus and the grammar of natural language develops only through the combination of the results from those two chapters.

A. A Formal Model of Grammar

Taking seriously Wittgenstein’s claim that mathematical numerical expressions are grammatical demands a clear understanding of Wittgenstein’s definition of ‘grammar’. At first, Wittgenstein’s notion of grammar does not seem to be that of common use. The absence of an explicit definition of grammar in his published writings makes it difficult to tackle Wittgenstein’s stance on such questions as ‘What is the relationship between mathematical calculi and the grammar of natural language?’ In the *Big Typescript*, Wittgenstein briefly offers an explanation of ‘grammar’ which lacks specificity. Remaining faithful to Wittgenstein requires sticking to the textual evidence in evaluating his grammatical claims. However, this commitment does not counter using a formal understanding of grammar when working within the limits of what Wittgenstein explicitly wrote about grammar in this period. The previous chapter provided a precise formalized theory of grammatical analysis fundamental for understanding its interactions with mathematics.

B. *Anwendung* and the Grammar of Natural Language

The third chapter, on the notion of *Anwendung*, explained the role calculations play in the solution of practical problems. For Wittgenstein, mathematics consists entirely of calculations [*Rechnungen*]. Calculations cannot solve anything but mathematical problems. Mathematical calculations are used all the time to solve practical problems. However, even though they are essential to the solution of some practical problems, mathematical calculations do not justify predictions about affairs outside the calculus to which they belong. Therefore, investigating the relationship between mathematical calculation and factual prediction offers more insight than asking about the relationship between mathematical and empirical propositions.

Calculations also provide the grammar of predictive statements. For example, dividing twelve by three in the arithmetic of natural numbers furnishes the grammar of the statement ‘if I have eleven apples, I can share them among three people in such a way that each is given three apples’. Calculations give solutions to practical problems just as they do mathematical ones: they provide a grammatical rule whose applications are propositions. These propositions may occur either inside or outside the calculus in internal and external applications of the rules that constitute a mathematical calculus.

Another important conclusion from the third chapter was that mathematical expressions, like numerals, have the same meaning in natural language as in pure mathematics. Their grammar is the same. In particular, arithmetic provides the grammar of cardinal numerals in natural language. It is part of the grammar of natural language. This fifth chapter tests this thesis within the formal framework developed in the previous chapter.

C. Mathematics as Grammar

This fifth chapter proves that a grammatical analysis applied to natural language produces propositions with a naturally mathematical interpretation. The notion of *Anwendung*, as developed in chapter 4, grounds Wittgenstein's thesis that the grammatical analysis of a portion of language which is the application of a mathematical calculus results in a grammar identical to the calculus. For Wittgenstein, a mathematical calculus has two different sorts of *Anwendung*: an internal one in its very own calculations and an external one in natural language (or another calculus). Reflecting this, this chapter grammatically analyzes both the internal and external *Anwendung* of elementary arithmetic.

For the first task, consider the arithmetical calculus as a language where the correct calculations are the acceptable expressions. Taking natural language as given – where the acceptable expressions are none other than the grammatically correct statements – is sufficient for the second task. In both cases, it not only produces adequate arithmetical propositions, but also demonstrates the adequacy of Wittgenstein's philosophical interpretation of such propositions. In other words, this shows, first, that mathematical equations result from the grammatical analysis of the traces the arithmetical operations left behind and, second, that such equations express a connection between the numbers and operations as grammatical categories.

The first stage of such demonstration is a formal grammatical study of the notion of natural number. The first step defines numbers as grammatical categories. Adhering to Wittgenstein's philosophical theory of numbers, they are defined as the results of particular additions. Every addition results in a single number. The second step shows that numbers so defined are in fact classes of interchangeable expressions. Contexts of intersubstitutability in the language of arithmetical calculation define grammatical categories. Finally, the concluding step shows that the traditional recursive definition of natural number is equivalent both to the definition of numbers as the result of additions and to the conception of numbers as grammatical equivalence classes of numerals.

This requires showing that, for every natural number in the traditional sense, the Wittgensteinean grammar of arithmetical calculus possesses a category containing only the appropriate numerals. It is also necessary to show that, if the induction principle holds for them, they form the smallest set of categories, closed under successor, including zero. This requires defining the system of cardinal numbers through grammatical analysis. This analysis shows that a unique grammatical category for 'one' exists at the base of the cardinal numerical system. It also defines the function of successor in grammatical terms and translates the induction principle to the grammatical vocabulary developed in the previous chapter.

II. A Grammatical Analysis of the Internal *Anwendung* of Arithmetic.

> Wherever in a calculus one number can be replaced by another . . . They are the same.
>
> WL §8 p. 218

A. Introduction

This section shows how internal *Anwendung* works within a calculus. The calculus in question is the subsystem of elementary arithmetic containing elementary additions of natural numbers in base ten notation. It applies last chapter's formal model of grammatical analysis to this subsystem of arithmetical calculation .

1. Calculi as Languages

The first phase consists of fitting the arithmetical subsystem in the previous chapter's definition of language. For this purpose, the digits one to ten and the addition and equal signs define the alphabet of the language. Particular additions, that is, numerals linked by the addition sign, play the role of calculation signs in this calculus, and equations constitute the acceptable strings. Additions, equations and numerals make up the vocabulary of the language.

2. A Grammatical System of Natural Numbers

Three different accounts of 'number' appear in Wittgenstein's writings during the early thirties. The following section provides, within the framework of grammar, an explicit formulation of such accounts and then demonstrates their equivalence. Wittgenstein explicitly offers the first two accounts, while the third reformulates the traditional definition of a system of natural numbers.

1. A natural number is the result of an arithmetic operation, in this case: addition.

2. A natural number is the grammatical-equivalence class of a numeral, the class of all words interchangeable with a numeral throughout the language.

3.The set of natural numbers is the smallest set to include zero and be closed under successor.

(1) is **Definition 4.2.3. Corollary 4.2.3.1.** demonstrates the implication from definitions (1) to (2.) Finally, **Theorem 4.4.4.** shows the equivalence between definitions (2) and (3).

3. The Grammar of Addition

This section shows how a grammatical analysis of Wittgenstein's sort produces correct mathematical equations. After all, language analysis starts with a definition which naturally includes the class of correct equations. It would not be surprising if they were also the result of such analysis. The grammatical analysis would amount to less than a sleight of hand. However, this is not so. It is true that the beginning material includes traces of correct calculations as ‘given’. Nevertheless, the mathematical equations are results in the meta-language. They are explicitly grammatical rules about numbers, where numbers are grammatical categories. For example, that $3 + 4 = 7$ is a theorem of the grammar means that adding expressions belonging to the grammatical categories ‘number three’ and ‘number four’ results in an expression belonging to the grammatical category ‘number seven’. Equations play a double role in the calculus. They are both acceptable configurations in the language, and grammatical rules of the calculus. The introduction of a meta-language serves the purpose of making this double role explicit.

This section is important for two reasons. On the one hand, it supports the original hypothesis that Wittgenstein’s grammatical method of analysis gives results with a natural mathematical interpretation. Also, it justifies this research’s interpretation of Wittgenstein’s philosophy of mathematics. Just as the first part demonstrates that numbers are grammatical categories of inter-substitutable numerals, the second part shows that arithmetical equations connect calculations – in this case, additions – with their results.

B. A Formal Grammatical Analysis of a Subsystem of the Arithmetic of Natural Numbers with Addition as its Single Operation

This section follows the following notational conventions: Capital Latin letters stand for grammatical categories. Bold Greek upper case letters stand for other sets of expressions. Lower case Latin letters stand for particular expressions of the language, except for italic ones. Italic lower case Latin letters stand for variables in the definition of categories through *lambda* abstraction. A cursive upper case letter *L* stands for the language structure. ˘ stands for the set of numbers. All variables range over expression *types* and sets thereof and do not refer to expression tokens at all.

1. The Calculus as Language

Definition 4.1.1 [digits]: Let Δ = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} be the set of *digits.*

Definition 4.1.2 [numerals]: Let **N** be the set of all *numerals* in base ten notation, finite sequences of digits not starting with a numeral zero.

Definition 4.1.3 [additions]: Let **A** be the set of *additions* as recursively defined on the base of the numerals so that **A** = { a∞‘+’∞b | a, b **A** **N** }. Notice that avoiding parentheses builds associativity into the language.

Definition 4.1.4 [numerical expression]: Any expression different from ‘+’ and ‘=’ is called *numerical.*

Proposition 4.1.4.1: Every addition is a numerical expression. ~

Proposition 4.1.4.2: Every numeral is a numerical expression. ~

Proposition 4.1.4.3: Every numerical expression is either a numeral or an addition. ~

Definition 4.1.4 [addition operator on numerical expressions]: For every two numerical expressions a and b, a + b = a∞‘+’∞b.

Proposition 4.1.4.1: For every two numerical expressions a and b, a + b **A**. ~

Definition 4.1.5 [language]: Define language *L* by the structure <∑, E, W>, where ∑ = Δ {+, =}is its alphabet, W is the class of traces of correct calculations, that is, equations such as ‘3 + 4 = 7’ or ‘2 + 2 = 4’,[1] and **E** = W ∑ **N** **A** is the set of *basic*

[1]. The acceptable strings are not true statements of arithmetic, but the final traces of correct additions. This means that expressions like ‘3 + 4 = 7’ are included, but ‘7 = 7’ and ‘3 + 4 = 4 + 3’ are not. Cf. Chapter 2, section III B.

words or *expressions* of the language. This definition satisfies the three conditions of Definition 1.1 in Chapter 4: (1) Σ is a finite, non-empty set, (2) W and E are subsets of the set of finite strings of such words such that (∑ W) E and (3) every member of E is a substring of some element of W. In other words, every number occurs in a true addition equation.

2. Grammatical Number Theory

Definition 4.2.1 [result operator]: Given an addition a **A**, define the category R(a) = λ*x* (a∞‘=’∞*x*) as the *result* of addition a.

Defintion 4.2.2 [number]: A *number* is any category of the form λ*x* (a∞‘=’∞*x*) where a **A.** In other words, a category is a number if it is the result of some addition.

Proposition 4.2.2.1: For any number category C and numeral n, if n C, then C = [n].

Proof: Suppose n, m C to show that n ~ m. n and m may occur either at the right or the left of the ‘=’ sign. For every addition a, a∞‘=’∞n W iff a∞‘=’∞m W. This guarantees that m may replace any occurrence of n at the right of ‘=’. Furthermore, let a be one of such additions. In every addition different that a itself, a is substitutable for n or m. Also, in any addition, n and m are substitutable for a. By transitivity, n is substitutable for m in any addition. In consequence, they are interchangeable at either side of ‘=’.

3. The Grammar of Addition

Definition 4.3.1 [addition category of a numeral]: For every numeral n, let the *addition category* of n, written A(n), be the category λ*x* (*x*∞‘=’∞n). This is the category of all additions – numerals linked by the addition sign ‘+’ – whose correct result is n.

Proposition 4.3.1.1: For every a in A(n), n R(a). ~

Lemma 4.3.2: For every pair of numerals n, m in **N**, if there is a number N such that n, m N, A(n) = A(m).

Proof**:** Let N be a number, such that n, m N. Also, let a A(n). From the previous corollary, n R(a). Hence, R(a) is a number such that n belongs to it. From theorem 4.2.3.1, n can only belong to one number, so N = R(a). Hence, m R(a). Since a A(n), this means that (a∞‘=’∞m) W, which implies that a A(m). ~

Definition 4.3.3 [addition category of a number]: For any number N, define the *addition category* A(N) as A(n) where n N. Hence, A(N) = A(n) for all n in **N**.

Theorem 4.3.4: For any addition a in **A**, A(R(a)) = [a].

Proof**:** Obviously, a A(R(a)). Suppose b A(R(a)) to show that b ~ a. Let C be any category C such that a C, to show that b C. Since a and b are additions – numerals linked through the plus sign, they occur only to the left of the ‘=’ sign. Without loosing generality, C is of the form λ*x* (c∞‘+’∞*x*∞‘+’∞d∞‘=’m) where c, d A and m N. This means that (c∞‘+’∞a∞‘+’∞d∞‘=’m) W. Since b A(R(a)), R(b) = R(a). In consequence, (c∞‘+’∞b∞‘+’∞d∞‘=’m) W. So b λ*x* (c∞‘+’∞*x*∞‘+’∞d∞‘=’m) = C. ~

Definition 4.3.5 [numerical category]: A category A is *numerical* if for all a and b in A, R(a) = R(b).

Definition 4.3.5.1 [result of a numerical category]: Given a numerical category A, define the *result* of A as R(A) = R(a∞‘+ 0’) for any a in A.

Definition 4.3.6 [addition of numerals]: Given two numerical expressions a and b, define the *addition* of a and b, written as A(a + b), as the category λ*x* (*x*∞‘=’∞c) , where c is an expression in R(a∞‘+’∞b).

Proposition 4.3.6.1: (a∞‘+’∞b) A(a + b). ~

Lemma 4.3.7: Let A and B be two numbers such that a A and b B. For every pair of expressions a’ A and b’ B, a’∞‘+’∞b’ belongs to the addition of a and b.

Proof. Since, as noted above, the only contexts in which numerals of the same number differ is inside numerals, a’ may substituted for a and b’ for b in their addition without affecting the result. ~

Lemma 4.3.8. Let A and B be two additions such that a A and b B. For every pair of expressions a’ A and b’ B, a’∞‘+’∞b’ belongs to A(a + b).

Proof**:** A(R(a) + R(b)) = A(a + b). A(R(A) + R(B)) = A(a + b).

A(R(a’) + R(b’)) = A(a + b). A(a’ + b’) = A(a + b). ~

Theorem 4.3.9: Let A and B be two numerical categories such that a A and b B. For every pair of expressions a’ A and b’ B, a’∞‘+’∞b’ belongs to A(a + b).

Proof**:** Grammatical categories include only numerals and additions. ~

Definition 4.3.10. [addition of categories]: Define the addition of numerical categories A and B, written A + B, as A(a + b) where a A and b B.

Lemma 4.3.11: R(R(a) + R(b)) = R(a + b).

Proof**:** The addition of two numerals is not a numeral, but an addition. Hence, the addition of two numbers is not a number either. However, the result of a numerical category is always a number. ~

Theorem 4.3.12: For every equation of the form a + b = c in W, c belongs to the number which is the result of the addition [a]+[b]. In other words, R([a] + [b]) = [c].

Proof**:** R(a + b) = R(c). Hence, R([a] + [b]) = [c]. ~

Note 4.3.13: Notice that, however, for every equation of the form a + b = c in W, [a] + [b] = [c] is not a theorem of this theory. In other words, the addition of two numbers is not another number. Arithmetical equations are not numerical identity statements. That is precisely the anticipated result from Wittgenstein’s analysis of equations, where the ‘=’ sign does not symbolize identity but a connection between calculation and result.

4. A Grammatical System of Natural Numbers

Definition 4.4.1 [zero]: Let 0 be the [Wittgensteinean] category λ*x* (‘0 + 0=’∞*x*).

Proposition 4.4.1.1: ‘0’   0. ~

Definition 4.4.2 [one]: Let 1 be the [Wittgensteinean] category λ*x* (‘0 + 1=’∞*x*).

Proposition 4.4.2.1: ‘1’   1. ~

Definition 4.4.3 [successor]: Define the *successor* function S defined over the numbers such that S(N) is the number λ*x* (n∞‘+’∞‘1’∞‘=’∞*x*) where n is any expression in N.

Proposition 4.4.3.1: S(0) = 1.

Proof: ‘0 + 1 = 1’   W. ~

Theorem 4.4.4: Let ˘ be the set of natural numbers. ˘ is the smallest set closed under S such that 0   ˘.

Proof**:** This theorem requires proving: (i) that 0 is an element of ˘, (ii) that ˘ is closed under successor, and (iii) ˘ is the smallest set with these two properties.

i) Prove that 0   ˘, that is, 0 is a number.

Proof of (i): ‘0 + 0 = 0’ W and ‘0 + 0’ **A.** By the definition of *number*, ‘0’ λ*x* (‘0 + 0 =’∞*x*) ˘. For every n in **N**, n 0 iff n λ*x* (‘0 + 0 =’∞*x*), so λ*x* (‘0 + 0 =’∞*x*) = 0 ˘.

ii) Prove that if N belongs to ˘, so does S(N).

Proof of (ii): Let N be a number, show that S(N) is also a number. Let n be any expression of number N. n∞‘+ 1’ **A**. Since S(N) is the category λ*x* (n∞‘+’∞‘1’∞‘=’∞*x*), it is of the form λ*x* (a∞’=’∞*x*) where a **A**. In consequence, S(N) is a number.

iii) Let N* be a set of categories such that 0 N* and N* is closed under successor. Prove that ˘ N*. In other words, prove that zero reaches every number N in ˘ by repeated applications of the successor function.

Proof of (iii): Since every number has the form λ*x* (a∞‘=’∞*x*) where a **A**, (iii) amounts to the proposition that, for all a in **A**, R(a) ˘. Considering only additions consisting of two numerals united by the ‘+’ sign eases the presentation of this proof. However, the proof applies to additions of any finite length, as well. This proof is an induction on **A**. Ordering **A** is necessary. Consider the following: ‘0 + 0’, ‘0 + 1’, ‘1 + 0’, . . . The category λ*x* (‘0 + 0=’∞*x*), belonging to ˘ bases this induction. Since, λ*x* (‘0 + 0=’∞*x*) = 0 ˘*, the base holds. Now, suppose as inductive hypothesis, that for every a* < a, R(a*) ˘*, to show that R(a) ˘*. Since a = b + c for some b, c in either **A** or **N**, R(a) = R(b + c). If b or c is a numeral, then its grammatical-equivalence class belongs to ˘. This is obvious, because the successor function constructs all numerals from zero. If it is an addition then it is before a in the ordering of **A**, so the inductive hypothesis applies to it. In either case, R(b) ˘* and R(c) ˘*. From Lemma 4.3.11, R(R(b) + R(c)) = R(b + c). Since R(b) ˘*, b = d + 1 for some addition d. Hence, R(a) = R(d + c + 1) = R(R(d + c) + 1). But (d + c) < a, so R(d + c) ˘*. In consequence, R(R(d + c) + 1) ˘*. Finally, since R(a) = R(R(d + c) + 1), R(a) ˘*. ~

C. Mathematical Induction

Besides showing the desired hypothesis, the previous formal analysis of arithmetical addition for natural numbers produced another important conclusion. Theorem 5.4.4. shows that Wittgenstein’s notion of numbers as grammatical categories form a system of natural numbers. This stands against those who argue that, during this period, Wittgenstein rejected the induction principle for arithmetic. However, this appraisal needs qualification. Wittgenstein rejected the existence of universal mathematical propositions. For him, mathematical propositions of the form $x(\Phi x)$, where Φ is a mathematical concept, are not about *all* the members of such mathematical category.[2] Mathematical inductions are not inductions in the sense this word has in natural science. They are calculations. In consequence, mathematical inductions do not prove universal properties of all members of mathematical concepts. Mathematical propositions, whose proof is an induction are not more general than those any other calculation proves. They cannot prove anything general about other calculus.

2. Chapter 2, on mathematical concepts, presented Wittgenstein’s argument for this heterodox view.

III. A Grammatical Analysis of the External Application of Arithmetic

> I want to say numbers can only be defined from propositional *forms*, independently of the question which propositions are true or false.
>
> PR §102

The analysis of Wittgenstein's notion of *Anwendung* in Chapter 3 concluded that mathematics is part of the grammar of natural language. When Wittgenstein says that mathematical propositions are grammatical rules, the expression 'grammatical rules' is not a metaphor. Mathematical rules are equal to obviously grammatical ones like 'adverbs qualify verbs and adjectives, but not nouns'.

Not surprisingly, Wittgenstein's claim has endured scorn and ridicule both from philosophers and linguists. Reactions include responses such as 'mathematicians are not grammarians' and 'there is no way that one may ever learn mathematics from reading the dictionary'. Many people are under the wrong impression that natural language grammar cannot tell different numbers apart, because all numerical expressions share the same grammar. They think that the substitution of numerical expressions *inter alia* does not jeopardize their grammatical correctness. These people think that, in general, saying 'I bought ten chars at the market yesterday' is as correct as saying 'I bought three chairs at the market yesterday', even though ten and three are different numbers. However, as this section will show, they are mistaken. Natural language provides for the grammatical distinction between numbers.

One of the most common philosophical arguments against mathematics being grammar is that mathematical entities challenge the accepted concept of grammatical categories. However, this final section proves the contrary. Numbers, or mathematical entities in general, are grammatical categories in precisely the same sense as adjective, noun, etc. Furthermore, reading the dictionary actually teaches mathematics.

A. A Numerical System.

> Was die Zahlen sind? – Die Bedeutung der Zahlzeichen; und die Untersuchung dieser Bedeutung ist die Untersuchung der Grammatik der Zahlzeichen.
>
> Wir suchen nicht nach einer Definition des Zahl-Begriff. Sondern versuchen eine Darlegung der Grammatik des Wortes “Zahl” und der Zahlwörter. [PG PT. II. Section IV §18 p. 630]
>
> What are numbers? – What numerals signify; an investigation of what they signify is an investigation of the grammar of numerals.
>
> What we are looking for is not a definition of the concept of number, but an exposition of the grammmar of the word “number” and of the numerals. [PG PT. II - IV §18 p. 321]

The first step translates the conventional definition of a numerical system into the grammatical vocabulary and shows how the cardinal numbers result from the application of Wittgenstein’s grammatical analysis from the early thirties, as formalized in the previous chapter.

1. ‘One’

According to ordinary English grammar, the word ‘one’ belongs to almost every major grammatical category: noun, pronoun, adjective and even verb. Mostly, it appears in the composition of nominal expressions (complex names) such as ‘one friend of mine’, ‘one fine dog’, etc. There, it functions just like the single indefinite article ‘a(n)’. Dictionaries often offer them as synonyms. For example, the *Webster's Revised Unabridged Dictionary*, (1998) gives as first definition of the word ‘one’: “The same word as the indefinite article ‘a’, ‘an’.” Interestingly enough, the *American Heritage Dictionary of the English Language* (1996) conceives this use of ‘one’ not as an article, but as an adjective. However, that is mistaken, because ‘one’ will not replace most other adjectives. It is only substitutable by the indefinite singular article a(n). Consider the aforementioned examples: ‘I'm just one player on the team’ is as correct as ‘I’m just a player on the team’ and ‘That is a fine dog’ is as grammatical as ‘That is one fine dog’. Nonetheless, this last example clearly shows that the role of ‘one’ in the nominal expression ‘one fine dog’ is completely different than that of the adjective ‘fine’. For one thing, without the adjective ‘fine’, the nominal expression does not change its grammatical status and the statement does not lose its grammatical correctness. Even though saying ‘That is one dog’ makes sense, saying ‘That is fine dog’ does not.[3]

Its singular article status prefixes it to a singular common noun (the sign of a concept or pseudo-concept) to create a nominal expression (the name of an object or a pseudo-object). For example, adding the word ‘one’ to the singular common noun ‘sailor in town’ results in the singular nominal expression ‘one sailor in town’. So far, it shares its grammatical role with other singular articles such as ‘the’, ‘a’ or ‘this’. However, since ‘one’ is an indefinite singular article, the resulting single nominal expression is an indefinite nominal expression. It designates a singular but indefinite object (or pseudo-object). The expression ‘the sailor in town’ is a definite name, while ‘one sailor in town’ is indefinite. In this sense, only the articles ‘a’ and ‘an’ can replace it, depending on the morphology (orthography) of the following word.

Nevertheless, when the word ‘one’ appears at the beginning of a singular nominal expression, sometimes it function as an adjective. Frequently, when the word ‘one’ expresses being the only individual of a specified or implied kind, it functions as an adjective. For example, in the complex nominal expressions ‘The one person I could marry’ or ‘The one horse that can win this race’, ‘one’ plays the role of an adjective. In these rare cases, it can

3. Frege uses a similar argument to show that, since ‘one’ is not an adjective, unity is not a property of objects.

replace the synonymous adjective ‘only’. The resulting expressions have the same grammatical status, such as ‘The only person I could marry’ or ‘The only horse that can win this race’. Notice, however, that the adjectival use of ‘one’ cannot replace every adjectival occurrence of ‘only’.[4] For example, the sentence ‘Robert Smithson was an only child’ makes sense, while ‘Robert Smithson was an one child’ does not. ‘Robert Smithson was one child’ is also grammatically correct, where ‘one’ plays the role of the article, not of an adjective. Furthermore, the adjective ‘only’ modifies plural as well as singular nouns. ‘One’ functions as an adjective also when used synonymously with ‘united’ or ‘undivided’ as in “The church is therefore one, though the members may be many” or, most uncommon, synonymously with ‘single’ or ‘unmarried’, as in “Men may counsel a woman to be one.” The same stipulations apply to these cases as to ‘only’, except that, as adjectives, they can also complement the verb ‘to be’ to form a predicate. In summary, the category of ‘one’ as an adjective, is the disjunction of the category ADJECTIVE and the disjunction of the categories [‘only’], [‘united’] and [‘single’].

The second most common use of ‘one’ in ordinary English is that of a pronoun. In those cases, it expresses either an indefinitely specified individual, as in ‘She visited one of her cousins’, or an unspecified individual, as in ‘The older one grows the more one likes indecency’. In this later case, the indefinite pronoun ‘anyone’ can replace it. Being a pronoun means functioning alone as a nominal expression. Any nominal expression, including another pronoun of the same type, can replace it. This rule justifies the grammatical correctness of statements such as ‘The older anyone grows the more one likes indecency’ and ‘The older Virginia Woolf grows the more she likes indecency’. Like a

4. Furthermore, the word ‘only’ can also play an adverbial role, which is not true of ‘one’. For example, saying ‘I lived only for your love’ makes sense, but not ‘I lived one for your love’.

proper name, the word ‘one’ can stand alone as a whole name. Unlike the adjective ‘one’, the pronoun ‘one’ has a plural form. The pronoun ‘ones’ is the plural of pronoun ‘one’.

A third use of ‘one’ is that of a noun. For the most part, the noun ‘one’ is a numeral, naming number one. In those cases, it is part of arithmetical statements: not part of the external application of arithmetic, but of its internal application.‘One’ is also a non-numerical common noun referring to a single person or thing: a unit, for example, ‘This is the one I like best’. In these cases, it is a common pronoun. Just as a proper pronoun may replace a noun or nominal expression, the word ‘one’ stands in place of a common noun. The word ‘one’, for example, may replace the common noun ‘dog’ in ‘This is the dog that bit me’ resulting in the grammatically correct sentence ‘This is the one that bit me’. It may even replace the whole complex common nominal expression ‘dog that bit me’ resulting in the sentence ‘This is the one’. This substitution requires a singular definite nominal expression containing the common noun. A singular definite pronoun such as ‘the’, ‘this’ or ‘that’ must precede it in a grammatically correct sentence.

If P(x∞y) is a well formed sentence such that (x∞y) is a nominal expression and x is a definite article, then P(x∞’one’) is also a well formed sentence. In consequence, this category is $\lambda y\ [P(x\infty y)\quad W\ \&\ x\quad SDA]$, where W is the category of well formed sentences and SDA is the category of singular definite articles. If the common noun occurs in a singular indefinite pronoun, and the indefinite article preceding the noun itself is ‘a’, ‘an’ or ‘one’, the rule does not apply. Articles ‘a’, ‘an’ or ‘one’ cannot precede the noun ‘one’, because the expressions ‘an one’, ‘a one’ and ‘one one’ do not make sense. If the common noun is part of a singular indefinite pronoun, and the indefinite article preceding the noun is ‘any’ or ‘some’, then ‘one’ fuses with them into the word ‘anyone’ or ‘someone’. Just as with the pronominal use of ‘one’, the common pronoun ‘ones’ is the plural of ‘one’.

'Ones' stands for common nouns in plural definite nominal expressions. As the grammatical correctness of 'This one bit me' derives from the correctness of 'This dog bit me', the correctness of 'These dogs bit me' entails the correctness of 'These ones bit me'.

Finally, *The American Heritage Dictionary of the English Language* includes the following usage note for 'one':

> Usage Note: When constructions headed by 'one' appear as the subject of a sentence or relative clause, there may be a question as to whether the verb should be singular or plural. Such a construction is exemplified in the sentence "One of every ten rotors was found defective." Although the plural were is sometimes used in such sentences, an earlier survey found that the singular was preferred by a large majority of the Usage Panel. Another problem is raised by constructions such as 'one of those people who' or its variants. In the sentence "The defeat turned out to be one of the most costly blows that were ever inflicted on our forces," most grammarians would hold that the plural were is correct, inasmuch as the subject of the verb is the plural noun 'blows'. However, constructions of this sort are often used with a singular verb even by the best writers. Note also that when the phrase containing one is introduced by the definite article, the verb in the relative clause must be singular: "He is the only one of the students who has (not have) already taken."[5]

Taking these uses into consideration,[6] it is possible to define the number one as the grammatical category to which the English expression 'one' belongs. This category is the intersection of the five uses considered before. Since two of its synonyms – 'only' and 'a' – have no grammatical categories in common, this intersection captures the specificity of category ['one']. This is sufficient to define this unique category as the base of the system of cardinal numbers.

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5. *The American Heritage® Dictionary of the English Language*, Third Edition. Houghton Mifflin Company, 1996, 1992.

6. English has a final, but obscure, use of the word 'one'. 'One' is also a transitive verb meaning 'to cause' 'to become one'; 'to gather into a single whole'; 'to unite'; 'to assimilate'. For example: "The rich folk that embraced and oned all their heart to treasure of the world" [Chaucer] *Webster's Revised Unabridged Dictionary*, © 1996, 1998 MICRA, Inc.

2. ‘Two’.

The English word ‘two’ has less diverse uses than ‘one’. It functions as noun, article and adjective. As expected, these uses of ‘two’ are analogous to those of ‘one’. Just as ‘one’, the most common use of ‘two’ is in the composition of nominal expressions such as ‘two friends of mine’, ‘two fine dogs’, etc. Substituting ‘one’ for ‘two’ in a singular nominal expression and changing the number of the common nominal expression from singular to plural produces an analogous plural nominal expression. For example, the substitution of ‘one’ for ‘two’ and the ‘chair’ for ‘chairs’ in ‘one broken chair’ results in ‘two broken chairs’. This does not mean that one nominal expression freely substitutes for the other. Saying ‘I'm just one player on the team’ makes sense, but ‘I’m just two players on the team’ does not. Singular and plural nominal expressions remain different grammatical categories. In the majority of these cases, ‘two’ works as a plural indefinite article for exactly two individuals. Placing ‘two’ before a common noun or nominal expression in the plural form results in a nominal expression. For example, preposing the common noun ‘sailors in town’ in its plural with the word ‘two’ produces the plural nominal expression ‘two sailors in town’. As ‘one’, the word ‘two’ also occurs in nominal expressions as an adjective. As the word ‘one’ expresses ‘being the only individual of a specified or implied kind’, the word ‘two’ expresses ‘being the two individuals of a specified or implied kind’. The complex nominal expressions ‘The two persons I could marry’ and ‘The two scientists that synthesized the protein’ are examples of this. When adjective ‘one’ is synonymous with ‘only’, substituting ‘two’ for ‘one’ and switching the common nouns and subordinate nominal clauses from singular to plural results in a grammatical correct plural nominal expression. This way, ‘The one person who lives in this house’ transforms into ‘The two persons who live in this house’. However, this transformation does not apply to the other adjectival uses of ‘one’. ‘Two’ cannot replace ‘one’, when meaning ‘united’ or ‘single’.

Most of the times, noun ‘two’ expresses number two. Just as numeral ‘one’, the use of numeral ‘two’ in arithmetical statements is part of arithmetic’ internal *Anwendung.* Noun ‘two’ also refers to exactly two persons or things, for example in ‘These are my favorite two’.[7] In these cases, it behaves like a common pronoun. Just as the noun ‘one’ may replace a singular common noun, ‘two’ may replace a plural common noun. For example, substituting the common noun ‘dogs’ in ‘These are the dogs that bit me’ for ‘two’ results in the grammatically correct sentence ‘These are the two that bit me’. ‘Two’ may even replace the whole complex nominal expression ‘dogs that bit me’ resulting in the sentence ‘These are the two’.[8] This substitution is permissible when the common noun is part of a plural nominal expression whose main article is not ‘two’. In other words, it may occur preceded either by a plural definite pronoun such as ‘the’, ‘these’ or ‘those’, or by an indefinite one such as ‘any’ or ‘some’. If P(x∞y) is a well-formed sentence such that (x∞y) is a nominal expression and x is a singular article different from ‘two’, then P(x∞’two’) is also a well-formed sentence. In consequence, the definition of this category is λ*y* [P(x∞*y*)   W & x   SA & x≠‘two’], where W is the category of well-formed sentences and SA is the category of singular definite articles.

c. From ‘one’ to ‘two’ and beyond: succession and induction.

An important feature results from analyzing the grammar of the English words ‘one’ and ‘two’. The major uses of these words are analogous, requiring only minor adjustments between singular and plural[9]. Once made the proper number arrangements, and except for

7. As a matter of fact, another nominal use of ‘two’ exists. ‘Two’ also names something with two parts, units, or members, especially a playing card, the face of a die, or a domino with two pips.

8. Notice that in the sentence ‘these are two’, ‘two’ behaves like an adjective.

9. ‘One’ easily substitutes‘two’, but not vice versa.

the cases where ‘two’ functions as a numeral, ‘one’ may replace ‘two’ any time it occurs in English. Furthermore, ‘two’ and the rest of the cardinal numerals, such as ‘three’, ‘five’ or ‘five hundred’ share these three basic uses: (*i*) plural indefinite article – as in ‘I ate three glazed doughnuts yesterday’, (*ii*) adjective – as in ‘These are my favorite ten jazz albums of all time’, and (*iii*) indefinite pronoun – as in ‘I’ll take six, please’. Nevertheless, to define the category of cardinal number, it is necessary to make sure that no other word have these same grammatical uses.

Cardinal numerals share their grammatical role of plural indeterminate articles with words such as ‘some’ or ‘many’[10] (but not with plural determinate articles such as ‘these’, those’ and ‘the’). Consider the context λ*x* (‘I know (that) C(*x*∞y), but cannot tell which’) where C(*x*∞y) is a well-formed sentence and y is a plural common noun. It determines the category of indeterminate article. ‘I know that five men in this island are married, but I cannot tell which’ makes sense, despite the nonsense of ‘I know that the men in this island are married, but I cannot tell which’ or ‘I know that those men in this island are married, but I cannot tell which’. However, this context does not define the category of cardinal numbers. ‘I know that some men in this island are married, but I cannot tell which’ and ‘I know that many men in this island are married, but I cannot tell which’ both make sense. Wittgenstein addresses this problem in Appendix 8 of the first part of the *Philosophical Grammar*, entitled “the concept ‘about’. The problem of the heap’ [*Der Begriff “ungefähr”. Problem des ‘Sandhaufens’*].

In that Appendix, Wittgenstein defines the grammatical category of cardinal numbers through a context similar to the following: λ*x* ‘there aren’t *x* apples on the table anymore, for I took *y*’. If ‘one’ substitutes for *y*, only cardinal numerals for numbers larger

10. In PG PT. II Section IV §18, Wittgenstein writes: “Es gibt auch ein Zahlensystem ‘1, 2, 3, 4, 5, viele’.” “There is also a system of numbers ‘1, 2, 3, 4, 5, many’.”

than one may replace *x*. The context λ*x* ‘there aren’t *x* apples on the table anymore, for I took one’ defines the category ˜ of cardinal number (larger than one). In general, the substitution is grammatically correct only if x > y. This already allows us to define ‘two’ and, in general, the rest of the cardinal numbers through their use as indeterminate articles. For example, the contexts λ*x* ‘there aren’t three apples on the table anymore, for I took *x* of them’ and λ*x* ‘I took *x* apples from the three on the table’ determine the category two = [‘two’]. Furthermore, the grammatical definition of the successor function is:

> S(n) = λ*x* ‘there aren’t *x* apples on the table anymore, for I took *n* of them’ *y* [‘there aren’t *y* apples on the table anymore, for I took *n*’ (‘there aren’t *y* apples on the table anymore, for I took *x*’ v *y*~*x*].

Including the following induction principle, would conclude the grammatical definition of numerical system:

C [C(‘one’) & n ˜ (C(n) C(Sn))] (C = ˜).

This principles says that every context such that ‘One’ can fill its blank spaces, and any numeral can replace its predecessor, determines the category of cardinal number. However, this principle is false. In strict sense, no other cardinal numeral can replace ‘one’ in any context. ‘One’ is singular, and the rest of the numerals are plural. Different interpretations of this grammatical fact exist. It is possible to say that ‘one’ is not a cardinal number in strict sense. However, some simple adjustments of number allow for ‘two’ to succeed ‘one’. Still ‘two’ may not replace ‘one’ in all the contexts, even making the necessary adjustments of number. For example, consider the context ‘The plate smashed in *n* pieces’.[11] Saying ‘The plate smashed in two pieces’ makes sense, but ‘The plate smashed in one piece’ does not. It would seem that ‘two’ is not a cardinal either. Similarly, only numerals above two satisfy

11. λ*x* (‘This rectangle consists of *x* parts’) PG Pt. II section IV 618 p. 638 (p. 324).

the context λn ('This figure has n sides'). It makes sense to say 'This figure has three sides', but not to say 'This figure has two sides'.

> Es hat keinen Sinn, von einem schwarzen Zweieck in weißen Kreis zu reden; und dieser Fall ist analog dem: es ist sinnlos zu sagen, das Viereck bestehe aus = Teilen (keinem Teil). Hier haben wir etwas, wie eine untere Gremze des Zuahlens, noch ehe wir die Eins erreichen. [PG PT. II Section IV §18 p. 640]
>
> It makes no sense to speak of a black two-sided figure in a white circle; this is analogous to its being senseless to say that the rectangle consists of 0 parts (no part). Here we have like a lower limit of counting before we reach the number one. [PG Pt. II Section IV § 18 p. 326]
>
> Und hier entsteht nun der Irrtum: Man meint, da man von der Zwei- un Vier-Teilung sprechen kann, kann man auch von der Zwe Drei-Teilung sprechen, geradeso, wie man 2, 3 und 4 Äpfel zählen kann. Aber die Drei-Teilung – wenn es sie guabe – gehört ja einer ganz anderen kategorie, einem ganz anderen System an als die Zwei- und Vier-Teilung. In dem System, in dem ich von Zwe- und Vier-Teilung sprechen kann, kann ich nicht von Drei-Teilung sprechen. Das sind logisch ganz verschiedene Gebilde. [PR Appendix II p. 321]
>
> And this is where the mistake occurs: people think, since we can talk of dividing into 2, into 4 parts, we can also talk of dividing into 3 parts, just as we can count 2, 3 and 4 apples. But trisection – if there were such a thing – would in fact belong to a completely different system, from bisection, quadrisection. In the system in which I talk of dividing into 2 and 4 parts I can't talk of dividing into 3 parts. These are completely differen logical structures. [PR Appendix II p. 334]

For Wittgenstein, this means that cardinal numbers have no unique single system. Instead a non-hierarchical motley of cardinal, numerical systems exists. Each countable concept determines its own numerical system with its own base. Apples start in one, parts in two, and sides of a geometrical figure in three. No system of cardinal numbers is primary. In consequence, asking whether the induction principle holds for the cardinal numbers does not make sense. The induction principle is not a result of Wittgenstein's grammatical analysis. However, this is not a flaw or weakness. The question is not whether the induction principle is true or false, but for what numerical systems it holds.

B. Calculation: the case of division.

Basic arithmetical equations result from the grammatical analysis of natural language. Consider the case of division among natural numbers as an example of arithmetical calculation.[12] The calculation ‘11 ÷ 3 = 3’ provides the grammar of the statement ‘if I have eleven apples and want to share them among some people so that each is given three apples, I can give three people their share with two apples remaining’. In general, ‘if I have *x* apples and want to share them among some people so that each is given *y* apples, I can give *z* people their share with *w* apples remaining’ is grammatically correct if *z* is the result of dividing *x* between *y*, and *w* is the remanent. Hence, λ*x, y, z, w* ‘if I have *x* apples and want to share them among some people so that each is given *y* apples, I can give *z* people their share with *w* apples remaining’ grammatically defines the four-place relation that *z* is the result of dividing *x* between *y*, and *w* is the remanent. In consequence, the context λ*x, y, z, w* (‘if I have *x* apples and want to share them among some people so that each is given *y* apples, I can give *z* people their share with *w* apples remaining’) in natural language determines the same category as the context

$$\lambda x, y, z, w \ \left(\ y \ / \begin{array}{c} w \\ \hline x \\ z \end{array} \ \right)$$

in the internal *Anwendung* of arithmetic.

Every calculation statement in the calculus provides the grammar for some statement in natural language. The abstraction of different numerical expressions from that sentence determine categories co-extensional with the calculation and result concepts in the original equation. In general, every grammatical context in the calculus has an analogous in natural

12. The example is not fortuitous. Wittgenstein illustrate the notion of external *Anwendung* with addition in §15 of section III of the second part of the *Philosophical Grammar*. The previous chapter analyzed this example in detail.

language. Both contexts determine the same category. This correspondence justifies the reproduction of the results of the grammatical analysis in section II for the external *Anwendung* of arithmetic.

IV. Conclusion: A False Dilemma

Interpretations of the thesis that mathematical propositions are grammatical diverge in one major respect. Wittgenstein scholars agree that mathematical propositions are grammatical rules for the construction or transformation of statements. However, they disagree on whether or not they are rules of mathematical or natural language. The basis of this disagreement is a false dilemma. Both interpretations are correct, but incomplete when considered in isolation.

It is tempting to approach Wittgenstein's idea that mathematics is grammar as an analogy. For Wittgenstein, images, metaphors and analogies are important sources of philosophical insight and confusion. However, Wittgtenstein does not use ‘grammar’ metaphorically. Mathematical propositions are not *like* grammatical rules. They *are* grammatical rules. Calculi are languages and calculations are expressions. The propositions of a calculus constitute its syntax, because they determine the calculations correctness. Calculation is the construction of linguistic expressions in accordance with the calculus grammar. Correct calculations are correct expressions in the calculus language. For example, the axioms and theorems of arithmetic constitute the syntax of arithmetic calculation. Similarly, the axioms of geometry are the syntactic rules of geometrical construction. In general, calculations within a mathematical theory are linguistic constructions in a language, with the theory’s axioms as the rules of its syntax.

Mathematics is also part of the grammar of natural language. This chapter has proved that arithmetic is the syntax of numerical expressions in natural language. The application of the proper grammatical analysis to cardinal numerical expressions in natural language results in a grammar with a natural arithmetical interpretation. It proved that cardinal numerical expressions belong to a unique grammatical category, and that the basic axioms of arithmetic determine their grammar.

For Wittgenstein, looking for an unique universal grammar underlying the whole of natural language and its uses is the biggest mistake of conventional grammarians. They ignore the inherent and essential multiplicity of language. Instead of a unique grammar for the whole of language, grammarians ought to look for the many grammars behind the many uses of language. Then, they will realize that some of these grammars are some of the most successful mathematical calculi. Euclidean geometry, for example, is the grammar of natural language *when used for the description of objects in the visual space.* Every mathematical calculus, not only arithmetic, constitute a portion of natural language grammar. For example, the axioms of Euclidean geometry comprise the syntax of natural language descriptions of objects in visual space.

Conceiving mathematics only as part of the syntax of natural language is also a mistake. For Wittgenstein, mathematical calculi regulate both a variety of mathematical activities, such as counting, measuring, adding, etc. and a part of the syntax of natural language. The dilemma between these two interpretations vanishes with the realization that mathematical language *is* a part of natural language. Mathematical calculi do not need to be performed in an artificial symbolic or diagrammatical language. They can be expressed in terms of apples, pebbles or whatever. This way they are incorporated into natural language. Arithmetic, like the rest of mathematics, is a subsystem inside natural language. Statements about adding apples or substracting pebbles *is* as much arithmetic as statements about adding or substracting numbers. This segment of natural language obeys mathematics, because it constitutes its grammar.

Furthermore, both dimensions of grammar are identical. For example, measuring is calculation because it is a technique for the construction of linguistic expressions. One does not exist without the other. As a linguistic technique, measuring obeys a syntax and, as a calculation, it obeys the rules of a calculus. Both systems of rules are identical. The syntax of calculation is always ultimately mathematical.

The results of this fifth chapter are twofold. On the one hand, it has formally shown that numerical calculi actually constitute grammatical systems in the sense of Wittgenstein. On the other hand, it also showed that they are the grammar of their segment of natural language. It has proved that if the object language contains the appropriate numerical expressions, the resulting grammar will include at least some rules with a natural mathematical interpretation. A grammatical analysis of using numerical expressions, both in calculation and in natural language, has yielded familiar theorems of arithmetic.