Chapter 7
Zahlangaben Revisited

I. Introduction

This chapter fits Wittgenstein's account of grammatical proposition within the framework of his philosophy of mathematics. Chapters 2 and 3 developed Wittgenstein's theory of mathematical propositions and calculation, while chapters 4, 5 and 6 worked on the formal demonstration of the grammatical nature of mathematical propositions. This chapter brings the result of this latter research to answer the questions about *Zahlangaben* left unanswered in the first two chapters. The first section offers a new formal approach to the notion of grammatical concept and relates it to that of 'system of propositions'. It also finishes the grammatical account of mathematical *Zahlangaben* started in Chapter 1. The philosophical and formal developments of the previous chapter allow for the completion of this explanation. The sum of these results substantiates Wittgenstein's understanding of mathematical *Zahlangaben* as grammatical propositions

II. Grammatical Concepts and Systems of Propositions

A. Systems of Propositions

At the completion of the *Tractatus,* Wittgenstein had isolated himself from academic philosophy, except for some conversation and correspondence with Ramsey[1] and a few members of the Vienna Circle. Ramsey's criticisms of his logical atomism partially inspired Wittgen-

1. Wittgenstein's conversation and correspondence with the mathematician Frank Ramsey, whom he befriended during the twenties and until Ramsey's premature death at the age of twenty six in January 1930, included many of Wittgenstein's post-*Tractatus* ideas on mathematics. Ramsey praised Wittgenstein's extensive work on the philosophy of mathematics during this period. Hans Sluga, introduction to *The Cambridge Companion to Wittgenstein.* Vol. 3, *From the* Tractatus *to* Remarks on the Foundations of Mathematics*: Wittgenstein on the philosophy of mathematics* (New York: Cambridge University Press, 1996) 18.

stein's return to professional philosophy. Wittgenstein's grammatical approach during this middle period is his response to this criticism, as the following passage, from F. Waismann's notes from Christmas, 1929, confirms:

> Ich habe einmal geschrieben: 'Der Satz ist wie ein Maßstab an die Wirklichkeit angelegt. Nur die äußersten Teilpunkte berühren den zu messenden Gegenstand'. Ich möchte jetzt lieber sagen: Ein 'Satzsystem' ist wie ein Maßstab an die Wirklichkeit angelegt. Ich meine damit folgendes: Wenn ich einen Maßtab an einem räumlichen Gegenstand anlege, so lege ich *alle Teilstriche* zu gleicher Zeit an. Nicht die einzelnen Teilstriche werden angelegt, sondern die ganze Skala. Weiß ich, daß der Gegestand bus zum Teilstrich 10 reicht, so weiß ich auch unmittelbar, daß er nicht bis zum Teilstrich 11, 12 usw. reicht. Die Aussagen, welche mir die Länge eines Gegenstandes beschrieben, bilden ein System, ein Satzsystem. Ein solches ganzes Satzsystem wird mit der Wirklichkeit verglichen, nicht ein einzelner Satz. Wenn ich z. B. sage, der und der Punkt im Gesichtsfeld ist *blau*, so weiß ich nicht nur das, sondern auch, daß der Punkt nicht gruun, nicht rot, nicht gelb usw. ist. Ich habe die ganze *Farbenskala* auf einmal angelegt. Das ist auch der Grund dafür, warum ein Punkt zu gleicher Zeit nicht verschiedene Farben haben kann; warum ein syntaktisches Verbot besteht, daß fx für mehr als einen Wert *x* wahr sein kann. Denn wenn ich ein Satz*system* an die Wirklichkeit anlege, so ist damit – genau wie beim räumlichen – schon gesagt, daß immer nur *ein* Sachverhalt bestehen kann, nie mehrere.
>
> Ich habe all das bei der Abfassung meiner Arbeit noch nicht gewußt und meinte damals, daß alles Schließen auf den Form der Tautologie beruhe. Ich hatte damals nicht gesehen, daß ein Schluß auch die Form haben kann: Ein Mensch ist 2 m groß, also ist er nicht 3 m groß. Das hängt damit zusammen, daß ich glaubte, die Elementarsätze müßten unabhängig sein; aus dem Bestehen eines Sachverhaltes könne man nicht auf das Nichtbestehen eines andern Schließen. Wenn aber meine jetzige Auffassung mit dem Satzsystem richtig ist, so ist sogar die Regel, daß man aus dem Bestehen eines Sachverhaltes auf das Nichtbestehen aller übrigen schließen kann, die durch das Satzsystem beschrieben werden. [PR Appendix II, p. 304]

> I once wrote: 'A proposition is laid like a yardstick against reality. Only the outermost tips of the graduation marks touch the object to be measured'. I should now prefer to say: a *system of propositions* is laid like a yardstick against reality. What I mean by this is: when I lay a yardstick against a spatial object, I apply *all the graduation marks simultaneously.* It's not the individual graduation marks that are applied, it's the whole scale. If I know that the object reaches up to the tenth graduation mark, I also know immediately that it doesn't reach the eleventh, twelfth, etc. the assertions telling me the length of an object form a system, a system of propositions. It's such a whole system which is compared with reality, not a single proposition. If, for instance, I say such and such a point in the visual field is *blue*, I not

> only know that, I also know that the point isn't green, isn't red isn't yellow etc. I have simultaneously applied the whole color scale. This is also the reason why a point can't have different colors simultaneously; why there is a syntactical rule against *fx* being true for more than one value of *x*. For if I apply a *system* of propositions to reality, that of itself already implies – as in the spatial case – that in every case only *one* state of affairs can obtain, never several.
>
> When I was working on my book I was still unaware of all this and thought then that every inference depended on the form of a tautology. I hadn't seen then that an inference can also be of the form: A man is 6 ft tall, therefore he isn't 7 ft. This is bound up with my then believing that elementary propositions had to be independent of one another: from the fact that one state of affairs obtained you couldn't infer another did not. But if my present conception of a system of propositions is right, then it's a rule that from the fact that one state of affairs obtains we can infer that all the others described by the system of propositions do not. [PR Appendix II p. 317]

In the second paragraph of this passage, Wittgenstein refers to the *Tractatus Logico-Philosophicus* as 'My book'. In 2.1512, he wrote that a proposition lays against reality like a ruler [*Es ist wie ein Maßstab an die Wirklichkeit angelegt*]. The passage also refers to 2.15121, where he added "Only the end points of the marks on the ruler touch the object being measured" [*Nur die äußeren Punkte der Teilstriche berühren den zu messenden Gegenstand*]. For the Wittgenstein of the early thirties, systems of propositions, not single propositions, work as rulers to measure reality.

> Die Sätze werden in diesem Falle noch ähnlicher Maßstäben, als ich früher geglaubt habe. - Das Stimmen eines Maßes schließt automatisch alle anderen aus. Ich sage automatisch: Wie alle Teilstriche auf einem Stab sind, so gehören die Sätze, die den Teilstrichen entsprechen, zusammen, und man kann nicht mit einem von ihnen messen, ohne zugleich mit allen andern zu messen. - Ich lege nicht den Satz als Maßstab an die Wirklichkeit an, sondern das System von Sätzen.
>
> Man könnte nun die Regel aufstellen, daß derselbe Maßstab in einem Satz nur einmal angelegt werden darf. Oder, daß die Teile, die verschiedenen Applikationen desselben Maßstabes entsprechen, zusammengefaßt werden müssen. [PR §82 p. 100]

> In which case, propositions turn out to be even more like yardsticks than I previously believed. – The fact that *one* measurement is right automatically excludes all others. I say automatically: just as all the graduation marks are on *one* rod, the propositions corresponding to the graduation marks

> similarly belong together, and we can't measure with one of them without simultaneously measuring with all the others. – It isn't a proposition which I put against reality as a yardstick, it's a *system* of propositions.
>
> We could now lay down the rule that the same yardstick may only be applied once in one proposition. Or that the parts corresponding to different applications of one yardstick should be collated. [PR 82 p. 110]

> Unsere Erkenntnis ist eben, daß wir es mit Maßstäben und nicht quasi mit isolierten Teilstrichen zu tun haben.
>
> Jede Aussage bestünde dann gleichsam im Einstellen einer Anzahl von Maßstäben, und das Einstellen eines Maßstabes auf zwei Teilstriche zugleich ist *unmöglich.* [PR §84 p. 112]

> What we have recognized is simply that we are dealing with yardsticks, and not in some fashion with isolated graduation marks.
>
> In that case every assertion would consist, as it were, in setting a number of scales (yardsticks) and it's *impossible* to set one scale simultaneously at two graduation marks. [PR §84 p. 112]

Wittgenstein developed the transition from elementary propositions to a system of propositions in more depth in section VIII of the *Philosophical Remarks*, where he writes:

> §83 Der Begriff des "Elementarsatzes" verliert jetzt überhaupt seine frühere Bedeutung.
>
> Die Regeln über "und", "oder", "nicht" etc., die ich durch die W-F-Notation dargestellt habe, sind ein Teil der Grammatik über diese Wörter, aber nicht die ganze.
>
> Der Begriff der unabhängigen Koordinaten der Beschreibung: Die Sätze, die z.B. durch "und" verbunden werden, sind nicht voneinander unabhängig, sondern sie bilden ein Bild und lassen sich auf ihre Vereinbarkeit oder Unvereinbarkeit prüfen.
>
> In meiner alten Auffassung der Elementarsätze gab es keine Bestimmung des Wertes einer Koordinate; obwohl meine Bemerkung, daß ein farbiger Körper in einem Farbenraum ist etc., mich direkt hätte dahin bringen können.
>
> Eine Koordinate der Wirklichkeit darf nur einmal bestimmt werden. [PR §83 p. 101]

> §83. The concept of an 'elementary proposition' now loses generally its earlier significance.
>
> The rules for 'and', 'or', 'not' etc., which I represented by means of the T-F notation are *a part* of the grammar of these words, but not *the whole.*
>
> The concept of independent coordinates of description: the propositions joined, e. g., by 'and' are not independent of one another, they form *one* picture and can be tested for their compatibility or incompatibility.

> In my old conception of an elementary proposition there was no determination of the value of a co-ordinate; although my remark that a coloured body is in a colour-space, etc., should have put me straight on to this.
>
> A co-ordinate of reality may only be determined *once*.
>
> If I wanted to represent the general standpoint I would say: 'You should not say now one thing and now another about the same matter'. Where the matter in question would be the coordinate to which I can give *one* value and no more. [PR §83 p. 111]

Instead of atomic and molecular propositions, reality is described through sets of coordinates. Ascribing an attribute to an object is fixing a coordinate of reality. Hence, it can only be determined *once*. Even *zero* or *nothing* remain a value. In the *Tractatus*, Wittgenstein considered each 'elementary proposition' as one single coordinate which could only take two possible values, *true* or *false*. He also considered elementary propositions logically independent from each other. By the early thirties, Wittgenstein had recognized the error in his previous view. In the middle period, Wittgenstein wanted to construe a complete logical space where propositions are like points in a geometrical space, related through geometric/grammatical laws. Logical space remains for the old logic, because *true* and *false* are still suitable values for certain coordinates. However, other coordinates need a new *logic*.

1. Systems of Propositions Formally Defined

A system of propositions is a set of propositions in the form Cx associated with a context λx (Cx). However, not all contexts yield systems of propositions. A system of propositions must also satisfy the further condition that one and only one of its propositions is true.

> Wie verhält es sich aber mit allen scheinbar ähnlichen Aussagen, wie: Ein materieller Punkt kann nur *eine* Geschwindigkeit auf einmal haben, in eine Punkt einer geladenen Oberfläche kann nur *eine* Spannung sein, in einem Punkt einer warmer Fläche nur *eine* Temperatur zu *einer* Zeit, in *einem* Punkt eines Dampfkessels nur *ein* Druck etc.? Niemand kann daran

> zweifeln, daß das alles Selbstverständlichkeit sind und die gegenteiligen Aussagen Widersprüche. [PR §81, p. 99]
>
> What about all assertions which appear to be similar, such as: a point mass can only have *one* velocity at a time, there can only be *one* charge at a point of an electrical field, at one point of a warm surface only *one* temperature at one time, at one point in a boiler only *one* pressure etc.? No one can doubt that these are all self-evident and that their denials are contradictions. [PR §81 p. 109]

Definition 5.1.1 [system of propositions]: Given a grammatical categories G and a context λ*x* (C(*x*)) in the language *L*, the set of expressions resulting from the substitution of expressions in G in context C, S = { C(e) | e G } is a *system of propositions* iff C G and !C(e) S which is true.

The propositions ‘This page is 1 in. wide’, ‘This page is 2 in. wide’, ‘This page is 3 in. wide’, ‘This page is 4 in. wide’, etc. form a system of propositions associated with the category of the width of this page, because one and only one of them obtains, in such way that,

(i) even though only one of them obtains, they are all possible, they all make sense,

(ii) that they are all false is impossible (saying ‘This page has no width’ does not make sense), and

(iii) if one of them obtains, the others are false.

Notice that two grammatical categories associated are at play in every system of propositions, and they are not necessarily the same. In the previous example, the context λ*x*(‘This page is’^*x*) does not yield the category ‘width of this page’. Consider, well-formed-sentences like ‘This page is white’ or ‘This page is my dissertation’s 165th’. Clearly, ‘white’ or ‘my dissertation’s 165th’ are not expressions of this page’s width. However, it is not necessary that the abstraction of one width expression produce the context associated to the system of propositions. It is sufficient that the width expressions satisfy the relevant context. In definition 5.1.1, that G is a subset of C guarantees that every substitution of expressions of G in C yield a well-formed-sentence.

2. Complete Descriptions

Definition 5.1.2 [complete description]: Every proposition in a system of propositions is a *complete description*.

Wittgenstein calls the propositions in a system "complete descriptions"[2]. They fully determine co-ordinates of reality by assigning them a value. The value assigned to an object's velocity of at a certain moment completely describes that velocity. It says everything about the velocity of that object at that time.[3]

> Wie ist es möglich, daß f(a) und f(b) einander widersprechen, wie es doch der Fall zu sein scheint? z.B., wenn ich sage, "hier ist jetzt rot" und "hier ist jetzt grün"?
>
> Es hängt das mit der Idee der vollständigen Beschreibung zusammen: "Der Fleck ist grün", beschreibt den Fleck vollständig, und es ist für eine andere Farbe kein Platz mehr.
>
> §77 How is it possible for *f(a)* and *f(b)* to contradict one another, as certainly seems to be the case? For instance, if I say 'there is red here and now' and 'there is green here now'?
>
> This is connected with the idea of a *complete description*: 'The patch is green' describes the patch completely, and there's no room left for another color. [PR §77 p. 106]

2. Wittgenstein uses the expression 'complete description' to emphasize the contrast with the *Tractatus* notion of 'complete analysis'. At that time, Wittgenstein believed that "a proposition isn't an elementary proposition unless its complete logical analysis shows that it isn't built out of other propositions by truth-functions." [PG Appendix I 4B p. 211] By the thirties, he had realized that his previous notion of logical analysis was senseless, and, in consequence, changed his definition of elementary proposition to one that "doesn't contain a truth-function and isn't defined by an expression which contains one" [PG Appendix I §4B p. 211]. Cf. also *Notizbücher* 06.17.1915.

3. Cf. *Zettel* Sct 311 ln 6: "Eine Rede vollständig (oder unvollständig) wiedergeben. Gehört dazu auch die Wiedergabe des Tonfalls, des Mienenspiels, der Echtheit oder Unechtheit der Gefühle, der Absichten des Redners, der Anstrengung des Redens? Ob das oder jenes für uns zur vollständigen Beschreibung gehört, wird vom Zweck der Beschreibung abhängen, davon, was der Empfänger mit der Beschreibung anfängt".

> Zu sagen, daß eine bestimmte Farbe jetzt an einem Ort ist, heißt diesen Ort vollständig beschreiben. [PR §80 p. 98]
>
> To say that a particular color is now in a place is to describe that place *completely*. [PR §80 p. 108]

Understand these definitions requires the introduction of a more definite definition of grammatical concepts.

3. Grammatical Concepts Formally Defined

Chapter 1 presented grammatical concepts as the logical product of its members – an extended disjunction. However, Wittgentein's notion of 'system of propositions' allows for a more precise presentation of grammatical concepts, evidencing their grammatical nature.

Definition 5.2.1 [semantic counterprart]: Concept S is a *semantic counterpart of* grammatical category C = λx (Px) iff for every expression e, S(e) iff e C = λx (Px). In other words, a concept S is the semantic counterpart of a grammatical category C iff it contains the meaning of those and only those expressions in category C.

Definition 5.2.2 [internal description]: Proposition S(e) is an internal description iff S is the semantic counterpart to a grammatical category C such that 'e' belongs to C.

Definition 5.2.3 [grammatical concept]: Concept C is a grammatical concept if and only if it is a grammatical category or the semantic counterpart of one.

B. Grammatical Concepts and Systems of Propositions

These definitions clarify Wittgenstein's examples in the previous passages from the *Philosophical Remarks*.. Consider the three examples Wittgenstein offers in the passage quoted above – length, height and color, and those in section VIII – velocity, temperature, electrical charge and pressure. They are all grammatical categories. The concept of velocity, for example, forms a system of propositions with the context resulting from abstracting the particular value in any velocity statement. The same holds for pressure, electrical charge and temperature. Take a physical statement that says that a certain point in an object has a given temperature at a fixed moment in time, for example, ‘The tip of my pencil was at 21˚C at the stroke of midnight on New Year’s Eve 2000’. Abstract the actual temperature predicated in the statement: 21˚C. Let C be the resulting context, λ*x* (‘The tip of my pencil was at *x* at the stroke of midnight on New Year’s Eve 2000’). All acceptable substitutions of temperature expressions expressions for *x* form the system of propositions: {‘The tip of my pencil was at 5˚K at the stroke of midnight on New Year’s Eve 2000’, ‘The tip of my pencil was at 0˚C at the stroke of midnight on New Year’s Eve 2000’, ‘The tip of my pencil was at 10˚C at the stroke of midnight on New Year’s Eve 2000’, . . .). Every proposition in the system express0es a possible state of affairs in the world. The temperature expressions that satisfy C – ‘0˚C’, ‘10˚C’, ‘5˚K’, ‘100˚K’, etc. – refer to different temperatures. Furthermore, that the tip of my pencil has one and only one temperature is syntactically necessary. Since temperature measures can be made in more than one scale (Kelvin, Centigrade, etc.), the substitution of a temperature expression for *x* in T resulting in a true proposition is not unique. However all the expressions that do express the same temperature.

Color is a grammatical concept, because if it makes sense to predicate a color from a point in the visual field at a certain moment, it must be of one and only one color. In other words, given any context resulting from the abstraction of a color term from a complete description, any replacement of the color term producing a true statement must refer to the same color. The same holds for height. Anything that has some height has one and only one height. The category ‘height’ forms a system of propositions with the context λ*x* (‘I am'∞*x*∞'tall'), because *x* only has one value. Every height expression that may replace *x*, resulting in a true statement, must express the same single height.[4]

This does not mean that optics, dynamics or thermodynamics may reduce to grammar, or that temperature, color and velocity are not actual physical concepts, but disguised linguistic ones. However, the scale used for measuring the value of these concepts is a grammatical convention indeed. The Centigrade and the Kelvin systems are yardsticks for the measurement of temperature. The scheme of colors for ascribing color to spots in visual space is also a grammatical yardstick.

> Die Einheitsstrecke gehört zum Symbolismus. Sie gehört zur Projektionsmethode. Ihre Länge ist willkürlich . . . Wenn ich also eine Strecke "3" nenne, so bezeichnet hier die 3 mit Hilfe der im Symbolismus vorausgesetzten Einheitsstrecke.
>
> Dasselbe kann man auch auf die Zeit anwenden. [PR §45 p. 69]
>
> The unit length is part of the symbolism. It belongs to the method of projection. Its length is arbitrary. . . And so if I call a length '3', the 3 signifies via the unit length presupposed in the symbolism.
>
> You can also apply these remarks to time. [PR §45 p. 79]

Predicating a grammatical concept is significantly different from assigning a value to a grammatical concept. A grammatical proposition predicates a grammatical concept. A genuine proposition assigns it a value. For example, saying that a point in visual space is colored is very different from saying which color it is. The first proposition is grammatical,

4. This idea committed Wittgenstein to claim that every calculation has one result and that mathematical propositions with different proofs are actually altogether different propositions. However, these consequences are not as absurd as they seem. Remember that Wittgenstein views calculations and proofs as rule-governed activities. That they have one single and unique result is not surprising. If a calculation did not arrive at any result, it would not be a calculation. Talking about calculations with more than one result is also nonsense. The obvious case of equations with more than one root is not a counterexample, because it actually hides a multiplicity of mathematical propositions. The different process arriving at each root is a different calculation. Furthermore, discovering that they are both roots of the equation requires yet another calculation. In either case, Wittgenstein's rule of syntactic necessity expresses the close connection between process and result.

while the latter is genuine. Whether or not a concept applies to a term is a grammatical issue. The concept's value is not. 'Color' is a grammatical concept. However, this does not mean that optics is nothing but grammar. This is true, not only of colors, but of other grammatical concepts. For example, determining whether a statement is a proposition is a grammatical calculation. Whether it is true or false is not.

1. A Warning Notice

It is important not to be too hasty at pairing categories with contexts to form systemns of propositions. Consider the following example. It is possible that, at noon, on my twenty first birthday, I murdered Mrs. Von Bulow. In other words, the statement ' I murdered Mrs. Von Bulow' is acceptable in ordinary English. The grammatical category λ*x* ('At noon, on my twenty first birthday, I murdered'∞*x*) contains proper names like 'Mrs. Von Bulow', 'Nezahualcoyotl', 'John F. Kennedy', etc., pronouns like 'you', 'myself', etc. and other expressions referring to persons. It would seem that the concept 'person' forms a system of propositions with the grammatical context λ*x* ('At noon, on my twenty first birthday, I murdered'∞*x*). For every expression e in λ*x* ('At noon, on my twenty first birthday, I murdered'∞*x*) , e∞'is a person' is true: Mrs. Von Bulow was a person, you are a person, I am a person myself, etc. However, this is not so. Otherwise, the statement 'At noon, on my twenty first birthday, I murdered one and only one person' would be necessary. From the possibility of me murdering someone, it would follow of necessity that I murdered one and only one person. However, that I murdered one and only one person is certainly not necessary.

Suppose that, on my twenty-first birthday, I threw a bomb at a train. The bomb exploded killing all of its passengers, some of them at the same time, exactly at noon.[5] This example shows that, at noon, on my twenty first birthday, I could have murdered more than one person. In that case, more than one expression in the category ‘person’ might have produces a true statement when substituted in the relevant context. If, by throwing that bomb at the train, I had killed Ana Mendietta, John F. Kennedy and Gustavo Colossio, then all the expressions ‘Ana Mendietta’, ‘John F. Kennedy and Gustavo Colossio’ and ‘Gustavo Colossio, Ana Mendietta and John F. Kennedy’ would belong to the category ‘person’ and would satisfy λx (‘At noon, on my twenty first birthday, I murdered’∞x) producing a true statement. In consequence, that I murdered one and only one person is certainly not necessary.

These two counterexamples show that the concept person does not form a system of propositions with grammatical category λx (‘At noon, on my twenty first birthday, I murdered’∞x).

III. *Zahlangaben* Revisited

Chapter 5 demonstrated the existence of a grammatical category in natural language corresponding to the mathematical concept of cardinal number. Genuine *Zahlangaben* form systems of propositions associated with this grammatical category. This category includes all cardinal numerical expressions, like ‘three’, ‘none’, ‘five hundred’, ‘three million’, etc. Since cardinal numbers express quantities, the concept ‘quantity’ is this category semantical counterpart. In consequence, *Zahlangaben* are complete cardinal descriptions, which form systems of propositions associated with ‘quantity’.

5. Thanks to Bo Ram Lee at Columbia University, for the example.

Consider a genuine *Zahlanagabe*, like ‘There are three men on this island’. It expresses a possible state of affairs in the world. The abstraction of the actual quantity of men predicated in the statement results in the context λx (‘There are x men on this island’). The system of expressions associated with ‘quantity’ and this context includes such propositions as ‘There are no men on this island’, ‘There are three men on this island’, and ‘There are three hundred men on this island’. In this system of propositions, the cardinal expressions depict the quantity of men on this island. Hence, there is only one quantity of men on this island.

Just as optics and thermodynamics do not reduce to grammar, quantities cannot reduce to grammar either. However, the system of cardinal numbers as a scale for measuring quantities is a grammatical convention. Just as with the Centigrade scale of temperature and the scheme of colors, the system of cardinal numbers is also a grammatical yardstick.

A. On the Extensionality of Mathematical Concepts

For Wittgenstein, mathematical concepts are not genuine concepts in the same sense as genuine concepts such as ‘spoon’, ‘chair’ or ‘person in this room’. Mathematical concepts are grammatical categories.[6] Since they are actually grammatical categories, their extensions completely determine them. In a certain sense, they are their own extensions. Mathematical and logical concepts such as unity, permutation, function, et cetera are not genuine concepts, because they lack proper intensionality. Unlike genuine concepts, the intension and the extension of these pseudo-concepts are the same.

> Ordinarily, the replacement of a function by a list (class) is mistaken. We say something different when we talk about a class given in extension and when we talk about a class given by a defining property. Intension and extension are not interchangeable. For example, it is not the same thing to say “I hate the man sitting in the chair” and “I hate Mr. Smith.” But it is

6. PR §116.

> otherwise in mathematics. In mathematics there is no difference between “the roots of the equation $x^2 + 2x + 1 = 0$” and the list [1 | 1],or between “the number satisfying $x + 2 = 4$” and “2.” The roots, and 2, are not described in the way the person is who satisfies the description “the man sitting in the chair.” [WL *Philosophy for Mathematicians* 1932-33 §1 p. 206]

For Wittgenstein, this means that they are actually disjunctions of their elements.[7] In paragraph 116 of the *Philosophical Remarks*, for example, he writes that “The permutations (without repetition) of *A*B *are A*B, B*A*. They are not the extension of a concept: they alone are the concept”.

> Gehen wir nun zur Schreibweise “(x) · fx” über, so ist klar, daß dies eine Sublimierung der Ausdrucksform unserer Sprache ist: “es gibt Menschen auf dieser Insel”, “Es gibt Sterne, die wir nicht sehen”. Und einem Satz “(x) · fx” soll nun immer ein Satz “fa” entscpreschen, und “a” soll ein Name sein. Man soll also sagen können: “(x) · fx, nämlich a und b” (“es gibt einen Wert von x, der fx befriedigt, nämlich a und b”), oder “(x) · fx, z. B., a”, etc. Und dies ist auch möglich in einem Falle wie: “es gibt Menschen auf dieser Insel, nämlich die Herrn A, B, C, D.” Aber ist es denn für den Sinn des Satzes “es gibt Menschen auf dieser Insel” wesentlich, daß wir sie benennen können, also ein bestimmtes Kriterium für die Identifizierung festlegen? Das ist es nur dann, wenn der Satz “(x) · fx” als eine Disjunktion von Sätzen von der Form “f(x)” definiert wird, wenn also z. B. festgelegt wird: “Es gibt Menscher in dieser Insel” heiße “Auf dieser Insel ist entweder Herr A oder B oder C oder D oder E”; wenn man also den begriff ‘Mensch’ als eine Extension bestimmt (was natürlich ganz gegen die normale Verwendung dieses Wortes wäre). (Dagegen bestimmt man z. B. Den Begriff “primäre Farbe” wirklich als eine Extension.) [PG Pt. I Appendix 2. p. 398]

> If we turn to the form of expression “(x) · fx” it’s clear that this is a sublimation of the form of expression in our language: “There are human being on this island”, “There are stars that we do not see”. To every proposition of the form “(x) · fx” there is supposed to correspond a proposition “fa”,

7. Extensionality is a property of all grammatical specification concepts, not only mathematical ones. For example, regarding the concept ‘pure color’, which is a grammatical category according to Wittgenstein, he writes in PR §116: “If I am right, there is no concept ‘pure colour’; the proposition '*A* has a pure colour' simply means '*A* is red, or yellow, or green, or blue'. 'This hat belongs to either *A* or *B* or *C*' isn't the same proposition as 'This hat belongs to someone in this room', even if as a matter of fact only *A*, *B* and *C* are in the room, since that needs saying.” [Wenn ich recht habe, so gibt es keinen Begriff ‘reine Farbe’; der Satz ‘A hat eine reine Farbe’ heißt einfach ‘A ist rot, oder gelb, oder blau, oder grün’. ‘Dieser Hut gehört entweder A oder B oder C’ ist nicht derselbe Satz wie ‘dieser Hut gehört einem Menschen in diesem Zimmer’, selbst wenn tatsächlich nur A, B, C in Zimmer sind, denn das muß erst dazugesagt werden.]

> and “a” is supposed to be a name. So one must be able to say “(x) · fx, namely a and b” (“There are some values of x, which satisfy fx, namely a and b”), or “(x) · fx, e. g. a”, etc. And this is indeed possible in a case like “There are human beings on this island, namely Messrs A, B, C, D.” But then is it essential to the sense of the sentence “There are men on this island” that we should be able to name them, and fix a particular criterion for their identification? That is only so in the case where the proposition “(x) · fx” is defined as a disjunction of propositions of the form “f(x)”, if e. g. it is laid down that “there are men on this island” means “Either Mr. A or Mr. B or Mr. C or Mr. D or Mr. E is on this island” – if, that is, one determines the concept “man” extensionally (which of course is quite contrary to the normal use of this word.) (On the other hand the concept “primary colour” really is determined extensionally.) [PG Pt. I, Appendix 2. Pp. 203, 204]

Knowing a mathematical concept involves knowing its extension. They are not *indefinite*.[8] For any grammatical category X, 'A is X' simply means 'A is x_1, or x_1, or x_1, or . . .' where x_1, x_2, x_3, . . . are *the* Xs[9]. Since grammatical categories are genuinely disjunctions, mathematical propositions have no generality. Non-mathematical *Zahlangabe*, on the contrary, are general. By *general* [allgemeine], Wittgenstein means that their logical form is a quantified proposition. For instance, ‘3 men are in this room’ has the form ‘(x, y, z) : x is a man and is in this room and y is a man and is in this room and z is a man and is in this room, etc.’[10] Being a man and being in this room are genuine concepts. They are not grammatical categories. Unlike mathematical ones, non-mathematical *Zahlangaben* contain an element of uncertainty [Unbestimmtheit][11]. For example, saying that ‘I know three men are in this room, but I do not know which’ makes sense.

8. PR §115

9. PR §116

10. PR §115

11. PR §115

> Es fällt auf, daß der Satz von den 3 Kreisen nicht die Allgemeinheit oder Unbestimmtheit hat, die ein Satz von der Form (*x*, *y*, *z*) *x*, *y*, *z) ·fx ·fy ·fz* besitzt. In diesem Fall kann man nämlich sagen: Ich weiß zwar daß 3 Dinge die Eigenschaft *f* haben, weiß aber nicht *welche*. Im Fall von den 3 Kreisen kann man das nicht sagen. [PR §115 p. 126]

> It is plain that the proposition about the three circles isn't general or indefinite in the way a proposition of the form (*x*, *y*, *z*) · f*x* · f*y* · f*z* is. That is, in such a case, you may say: Certainly I know that three things have the property f, but I don't know *which*; and you can't say this in the case of three circles. [PR §115 p. 136]

> Es hat also auf den Satz "(x) · fx" nicht in allen Fällen die Frage einen Sinn, "*welche* x befriedigen f" [PG Pt. I Appendix 2. p. 398]

> So it does not always makes sense when presented with a proposition "(x) · fx" to ask "*Which* xs satisfy f?" [PG Pt. I Appendix 2. p. 204]

In mathematics, knowing 'how many' implies knowing 'which'.[12]

In arithmetic, the question 'How many *X*s are such that *Y*' only makes sense in the case where *X* is the specification concept 'unit' and *Y* is a calculation concept. In consequence, it makes sense to ask how many units are in 3 + 4, but not how many numbers or additions are there in arithmetic.

> Begriffswörter in der Mathematik: Primzahl, Kardinalzahl, etc. Es scheint darum inmittelbar Sinn zu haben, wenn gefragt wird: "Wieviel Primzahlen gibt es?" ("Es glaubt der Mensch, wenn er nur Worte hört, . . .") In Wirklichkeit ist diese Wortzusammenstellung einstweilen Unsinn; bis für sie eine besondere Syntax gegeben wurde. [PG PT. II §24 p. 738]

> In mathematics there are concepts words: cardinal number, prime number, etc. That is why it seems to make sense straight off if we ask "how many prime numbers are there?" (Human beings believe, if only they hear words . . .) In reality this combination of words is so far nonsense; until it's given a special syntax. [PG Pt. II §24 p. 375]

It is very different to ascribe a cardinal number to a mathematical grammatical category than to a genuine concept. Counting objects which fall under genuine concepts, like apples on a table or shirts in a chest is radically different from counting mathe-

12. PR §115

matical entities, like circles on a geometrical plane or prime numbers in a class. Counting the members of genuine concepts exclusively results in genuine numerical propositions, not mathematical ones. Accordingly, it is not a calculation. Counting mathematical entities is.

If cardinality were a concept's property, as Frege professed, then ascribing a cardinal number to a mathematical concept would be describing it. However, mathematical *Zahlangaben* are not descriptions of this sort.[13] They are what Wittgenstein called *internal* descriptions.[14] Cardinality adds nothing to a mathematical concept.[15] The mathematical concept already includes its own cardinality. Consequently, the question *How many?* becomes "a straightforward problem."

> Wenn ich jemande frage, 'Wieviele Primzahlen gibt es zwischen 10 und 20?', so wird es sagen ich weiß es nicht im Augenblick, aber ich kann es jederzeit feststellen. Denn ist ja gleichsam schon irgendwo aufgeschrieben. [PR §114 p. 124]

> If I ask someone, How many primes are there between 10 and 20?, he may reply, 'I don't know straight off, but I can work it out any time you like.' For it's as if there were somewhere where it was already written out. [PR §114 p. 134]

13. In strict sense, for Wittgenstein, mathematical *Zahlangaben* are not *external* descriptions, but *internal* ones. However, internal descriptions do not ascribe properties to objects. Hence, mathematical *Zahlangaben* –as internal descriptions– do not ascribe cardinality as a property to concepts as objects. Unless one wants to introduce the external/internal distinction to the realm of properties. Then, it would make sense to say that cardinality is an external property of genuine concepts and an internal property of grammatical and mathematical concepts. However, whatever one may win in with such talk is questionable.

14. On the notion of 'internal description' and Wittgenstein's use of the distinction between internal and external descriptions to make sense of the distinction between mathematical and genuine *Zahlangaben*, cf. Chapter 1, part II, section E: "Mathematical Objects and Concepts."

15. Wittgenstein went as far as to suggest that the question *how many* only made sense of genuine concepts, not mathematical (or gramatical) ones. In *Philosophical Remarks* §99, Wittgenstein writes, "For instance, does it mean anything to say '*a* and *b* and *c* are three objects'? I think obviously not." It is hard to phantom Wittgenstein's point in this respect, except stressing the drastic differences between mathematical and genuine *Zahlangaben*.

B. Arithmetical Propositions

One finds three different characterizations of arithmetical statements in Wittgenstein's writings of the early thirties: as calculation statements, specification statements, and mathematical *Zahlangabe*. However, these are not three different kinds of statements, but three different approaches to arithmetical propositions. In Wittgenstein, there is a grammatical unity, not only to arithmetical, but to all mathematical statements. In the end, these three approaches collapse in one single grammatical kind.

Calculation statements connect calculations with their results. Equations and logical theorems are paradigmatic examples of calculation statements. Constitutive statements are statements of the form '*a* is a *B*' where *a* and *B* are mathematical categories. Mathematical *Zahlangaben* are statements of the form 'there are *n* *A*s', where *n* is a cardinal number and *A* is a mathematical concept. These seem to be different sort of arithmetical propositions because each seems to be true in a different sense. Calculation statements are true if and only if they connect a calculation with its correct result. Constitutive statements of the form '*a* is a *B*' are true if and only if the category *a* is included in *B*, that is, if every expression in *a* is included in *B*. Finally, a mathematical *Zahlangabe* of the form 'there are *n* *A*s', is true if the display of concept *A* presents *n* elements. However, despite their superficial differences, all mathematical statements share the same grammatical features.

Mathematical *Zahlangaben* are calculation statements, because counting mathematical entities, unlike counting genuine objects, is a calculation. A mathematical *Zahlangabe* of the form 'There are 'n' *A*s' is actually a calculation statement. It connects the calculation 'counting the *A*s' with its result 'n'. It says that counting the *A*s results in 'n'. Calculation statements are also constitutive statements. Since every calculation concept 'C' is a grammatical category, the concept 'being the result of C' is one, too. In consequence, the calculation statement connecting calculation 'C' with result 'a' is equal to the constitutive statement 'a is B' where 'B' is the concept 'being the result of C'. Consequently, all mathematical statements are constitutive statements. They all express grammatical relations between categories. All mathematical statements are 'internal descriptions'.

Mathematics is calculation, because it involves nothing more than displaying the extension of mathematical concepts. Mathematics is grammar, because mathematical concepts are grammatical categories. Displaying their extension is constructing it in obedience of the calculus' grammatical rules. Checking if such a display produces a given expression type, i.e. if it obeys the rules that define the type, is all it takes to determine the truth of a mathematical propositions. For example, checking if an arithmetical equation of the form '$C = a$' is true consists in (*i*) displaying the extension of the concept 'result of *C*', and (*ii*) checking if the extension includes *a*. Displaying the extension of 'result of *C*' is calculating *C*. Checking if constitutive statement '5 is a natural number' is true involves displaying the (extension of) number five and checking whether such construction follows the rules that define natural numbers.[16] This process is grammatical, because the criteria for displaying the extension of a mathematical concept is grammatical. For example, the rules of addition constitute the criteria for displaying the extension of the calculation concept 'addition'. In every case, performing a mathematical calculation equals producing expression tokens of a determined type. In consequence, every constitutive statement is also a calculation statement, where displaying the relevant, mathematical concept is the calculation. A constitutive statement of the form '*a* is a *B*' is a calculation statement connecting a calculation - the

[16]. Wittgenstein's phrasing of this process is misleading, for displaying the extension of a concept is not listing all of its memebers.Calculating whether 5 is a number or not is not displaying the extension of a mathematical category – 'natural number' – to see if numeral 5 appears in that extension. It does not consist in, first, producing the infinite collection of natural numbers and, then, check them one-by-one to see if 5 is one of them. It

display of the extension of concept *B* – with its result. The apparent diversity behind mathematics dissolves. All mathematics is calculation. All calculation is grammatical.

IV. Conclusions

In 2.1512 of the *Tractatus*, Wittgenstein wrote that a proposition is laid against reality like a ruler. In the early thirties, Wittgenstein wrote that systems of propositions, not single propositions, work as rulers to measure reality. Instead of atomic and molecular propositions, sets of coordinates describe reality. Ascribing an attribute to an object fixes a coordinate of reality. Hence, it can only be determined *once*. Assigning a value to a coordinate of reality fully determines it.

Given a grammatical category G and a context C, the set {C(e) | e   G} is a system of propositions if even though only one of the propositions in the system is the case, they are all possible; i. e. they all make sense. In other words, every predication of a grammatical concept has one and only one value. The measuring scale for the value of these concepts is a grammatical convention. The system of cardinal numbers is a scale for calculating the value of the grammatical concept ‘quantity’. In consequence, it is a grammatical convention.

Count objects which fall under genuine concepts is radically different from counting mathematical entities. Mathematical *Zahlangaben* are calculation statements, because counting mathematical entities, unlike counting genuine objects, is a calculation. Ascribing cardinality to a mathematical concept does not describe any of its properties, as Frege professed, but displays its extension.

A mathematical *Zahlangabe* of the form ‘There are *n* *A*s’ is a calculation statement, expressing that *n* is the result of counting the *A*s. Calculation statements, in turn, are a special case of constitutive statements. Since a calculation concept *C* is a grammatical category, the concept 'result of *C*' is, too. In consequence, a calculation statement, connecting a calculation *C* with its result *a*, equals a constitutive statement of the form '*a* is *B*' where *B* is the concept 'being the result of *C*'. In conclusion, all mathematical propositions are constitutive propositions expressing grammatical relations between categories

Mathematics consists entirely of displaying mathematical concepts' extensions. Calculating is following the grammatical rules determining the extension of a mathematical concept. Checking the truth of a calculation proposition 'C = a' is checking if the displayed extension of concept 'result of C' includes an expression of type 'a'. Displaying the extension of 'result of *C*' is performing calculation 'C'. Checking the truth of constitutive statement 'a is a C' is displaying an expression of type 'a' and checking whether such process obeys the rules that define the extension of'C'. This process is grammatical, because the criteria for displaying the extension of a mathematical concept is grammatical, too. In every case, performing a mathematical calculation is producing expression tokens of a determined grammatical type. In consequence, every constitutive statement is, in a certain sense, also a calculation statement where the calculation displays the relevant, mathematical concept's extension. A constitutive statement of the form '*a* is a *B*' is a calculation statement connecting a calculation – the display of the extension of concept *B* – with its result. In consequence, all mathematical propositions are calculations, and all calculation is grammatical.