Chapter 8
Grammatical Necessity

I. Introduction

Despite the 'linguistic turn' in philosophy at the beginning of the twentieth century, philosophers insist on stressing the boundaries between linguistics and their discipline, instead of taking advantage of their overlap. Philosophers of all sorts are reluctant to recognize the relevance of linguistic studies for their field. Even philosophers of language like Gilbert Ryle[1] and Stanley Cavell[2] have claimed that the results of linguistic science offer nothing to philosophy. This view results in the common assumption that when a philosopher like Wittgenstein talks about 'grammar' or 'syntax', he cannot refer to the linguistic disciplines of the same name. He must refer to some esoteric *logical* syntax or *deep* grammar. This dissertation aims at dispelling this common misconception. When referring to mathematical propositions as grammatical, Wittgenstein does not use the term 'grammatical' in a radically different way than linguists.

According to the maxim, *valet illatio ab esse ad posse*: The best way to show that something is possible is by doing it. In this case, showing how grammatical analysis can yield mathematical results is the best way to demonstrate that mathematics may be grammar. Chapter 6 pursued this goal. It developed familiar theorems of arithmetic out of a grammatical analysis of the use of numerical expressions, both in calculation and in natural language. Its results showed formally that numerical calculi not only constitute grammatical systems, but that they belong to the grammar of ordinary language. It proved that the

1. Gilbert Ryle, "Ordinary Language," *Philosophical Review* LXII (1953) : 167-186. "Use, Usage and Meaning," *Proceedings of the Aristotelian Society* supp. vol. XXV (1961) : 223-230.

2. Stanley Cavell, "Must we mean what we say?" *Inquiry* 1 (1958) : 172-212. "The availability of Wittgenstein's later Philosophy" *Philosophical Review* LXXI (1962) : 67-93.

grammar of any language with numerical expressions must include rules with a natural mathematical interpretation. Yet, to make the case that grammatical analysis, in fact, supports mathematical conclusions, responding to the various criticisms in the secondary literature is also necessary.

The present chapter defends Wittgenstein's position against criticisms. The first part focuses on some general arguments against the grammatical nature of mathematical propositions. These arguments support the claim that grammatical propositions, unlike mathematical ones, describe the usage of words. These arguments conclude that grammatical propositions are contingent, while mathematical ones are necessary. The first part of this chapter presents a defense of Wittgenstein's grammatical account of mathematical against the aforementioned objections. This defense is based on Morris Lazerowitz's "Necessity and Language", Zeno Vendler's "Linguistics and the *a-priori*" W. E. Kennick's "Philosophy as Grammar"[3], and J. Michael Young's "Kant on the Construction of Arithmetical Concepts."[4] The argument shows that the objections raised against Wittgenstein equivocate on the meaning of the adjective 'grammatical'. The second part of the chapter deals in closer detail with the Quine/Carnap debate and its relevance to Wittgenstein's grammatical account of mathematics during the early thirties.

3. W. E. Kennick, "Philosophy as Grammar" in G. E. Anscombe & M. Lazerowitz: *Ludwig Wittgenstein, Philosophy and Language* (Bristol: Thoemmes Press, 1996)

4. J. Michael Young, "Kant on the Construction of Mathematical Concepts" *Kant-Studien* 73 (1982): 17-46. Michael Young's article differs from those of Vendler, Lazerowitz and Kennick's in that it focuses on questions in the philosophy of mathematics. Michael Young uses an argument similar to the present one to "show that Kant is right in thinking that to ground *a priori* judgements, at least in arithmetic, upon ostensive constructions" is possible [p. 17] Primarily, Kant's and Wttgenstein's position regarding the construction of arithmetical concepts differ, because the rules of calculation that Kant refers to as 'the universal conditions of construction', are distinct from the concepts whose constructions they govern. Cf. Ibid. 28, 29.

II. On the Grammatical Nature of Mathematics

To understand the grammatical nature of mathematical propositions, it would be helpful to translate the formal results from the previous chapters into a more informal comparison between mathematical and more obvious grammatical propositions. This comparison will take place in two parts. The first part will develop the different senses in which statements are said to be 'grammatical'. The second part will make an analogy between these grammatical statements and properly mathematical ones. This analogy's goal is to clarify the sense in which mathematical propositions are grammatical. It also sets the basis to discuss the four general arguments against the grammatical nature of mathematics.

A. Arguments against the Grammatical Nature of Mathematics

In general, four major arguments are raised against the claim that mathematical propositions are grammatical:

1. Linguistic practice is an empirical fact. Hence, grammatical propositions about verbal usage are empirical generalizations and, consequently, not necessary. In contrast, mathematical propositions are necessary.

2. Understanding grammatical propositions as those that describe the usage of words implies that grammatical propositions are not necessary. Negating a true proposition about verbal usage is not a contradiction, but a false proposition.[5] Morris Lazerowitz , presents this objection as follows:

> The negation of a true verbal proposition is a false verbal proposition, but not a proposition which could not, in principle, be true. . . To use an expression of Wittgenstein's, we know what it would be like for a verbal *proposition*, which happens to be true, to be false. By contrast we do not know what it would be like for a false arithmetical proposition to be true, for example, for 4 + 3 to be less than 7.[6]

[5]. Morris Lazerowitz, introduction to M. Lazerowitz and Alice Ambrose eds. *Essays in the Unknown Wittgenstein* (Buffalo: Prometheus Books, 1984), 6.

[6]. Morris Lazerowitz: "Necessity and Language" (Lazerowitz 1984, 235)

3. If grammatical propositions record the usage of words, they must describe particular words in a particular language. Mathematical propositions do not, in general, say anything about vocabulary. Furthermore, if grammatical statements are about words, morphemes, etc. and their uses, then their truth depends on the existence of these linguistic entities. If, as Wittgenstein contended, '3 + 4 = 7' does not deal with abstract entities called numbers 3, 4 and 7, but with numeral types '3', '4' and '7', then it features a commitment to the existence of these numerals.

4. Finally, mathematical truths do not depend on the language expressing them. Hence, mathematical propositions cannot be grammatical. The peculiarities of one language are not sufficient to solve genuine mathematical problems.

The objection that grammatical propositions, unlike mathematical ones, are language-specific is at least as old as Moore's notes on Wittgenstein's lectures of Lent and May terms of 1930. He reports that Wittgenstein stated,

> . . . the proposition 'red is a primary color' was a proposition about the word 'red'[7]

Immediately after, Moore observed that,

> . . . if he had seriously held this, he might have held similarly that the proposition or rule '3 + 3 = 6' was merely a proposition or rule about the particular expressions '3 + 3' and '6'.[8]

Moore himself recognized the absurdities that his interpretation of Wittgenstein implied, when he commented,

> . . . he cannot have held seriously either of these views, because the *same* proposition which is expressed by the words 'red is a primary color' can be expressed in French or German by words which say nothing about the English word 'red'; and similarly the *same* proposition or rule which is expressed by '3 + 3 = 6' was undoubtedly expressed in Attic Greek and in Latin by words which say nothing about the numerals '3' and '6'. And

7. *Philosophical Papers*, 275 quoted in (Lazerowitz 1984, 16-17)

8. Ibid.

this was in fact what he seemed to be admitting in the passage at the end of (I).[9]

Mathematical propositions do not say anything about vocabulary. Furthermore, their truth does not depend on the language expressing them. For Moore, this meant that they cannot be grammatical.

Grammatical statements express language rules, even if they do not mention any explicitly linguistic entities like morphemes, words, etc. Still, mathematical propositions are categorical in their necessity. The following sections deal with this apparent tension.

B. Grammatical Statements

Consider an obviously grammatical transition of ordinary English language: the transition from passive to active forms. This transition may easily be formulated as a syntactic rule:

(1) The passive form of an active sentence a∞B∞c∞d (where a is the sentence's subject, B its verb, c the verb's direct compliment, and d is the string of indirect compliments of B) is the string c∞BE(c/B)∞PP(B)∞'by'∞a∞d, where BE(c/B) is the conjugation of the verb 'to be' in the number of c and the time of B, and PP(B) is the past participle form of verb B.

This rule allows us to transform active sentence (2) into passive sentence (3):

(2) Many persons have attended the dance marathon since its inception.

(3) The dance marathon has been attended by many persons since its inception.

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9. *Philosophical Papers*, 41 quoted in (Lazerowitz 1984, 17). Leaving aside for a moment the possibility that Wittgenstein might have actually said that 'red is a primary color' is *about* the word 'red', Moore's assumption that, if a grammatical proposition is about words, it must be about the words that occur in it is surprising. In many cases, grammatical propositions address the correct use of terms not in them. 'Number words can function as adjectives' is a grammatical proposition about the correct use of number words. Still, no number words occur in it. On the other hand, ''Spanish' is spelled with capital 'S'' is about the spelling of the word 'Spanish' in it. In general, external, grammatical statements are about the use of words in them, while internal ones are not.

This transformation may also be expressed in a single sentence (just like the conditionalization of a *modus ponens*):

(4) If many persons have attended the dance marathon since its inception, then the dance marathon has been attended by many persons since its inception.

This illustrates the double nature of grammatical application. For Wittgenstein, grammatical rules may be applied both in the formation (as in 4) and transformation (as from 2 to 3) of acceptable strings.[10]

Finally, the following statement also expresses this application of the rule expressed in (1):

(5) The passive form of "Many persons have attended the dance marathon since its inception" is "The dance marathon has been attended by many persons since its inception."

Since the rule for the transformation from active to passive is grammatical, statements (1), (4) and (5), and the transition between statements (2) and (3) may correctly be called grammatical. However, they are grammatical in a different sense. Their relation to the grammatical rules is different. Both (4) and the transition from (2) to (3) are *Anwendung* of the grammatical rule. Statements (1) and (5), on the contrary, are expressions of the rule. As such, they are *about* the grammatical rule. In consequence, they are *heteronomous* grammatical statements. Statement (4), in contrast, is a necessary *autonomous* grammatical statement. When Wittgenstein talks about grammatical statements, he is mostly referring to statements like (4), that is autonomous grammatical statements which do not express or are about any grammatical rule, but display it in its application.

This difference becomes essential once questions of truth and necessity come into play. It is clear that the question of truth can only be brought about statements and not about

10. Hence, he does not make a distinction between formation and transformation rules, as most traditional grammarians do.

transitions like that between (2) and (3). In those cases, the question of grammatical necessity is not that of necessary truth, but necessary transition. This difference will become essential when dealing with criticisms of grammatical necessity, like that of Quine. At the moment, this section will center on statements like (1), (4) and (5).

The most obvious criticisms to the necessary nature of grammatical statements focus on statements like (1), also called *external*[11] or *explicitly*[12] grammatical statements. These statements, as descriptions of grammatical rules, are language-dependent and contain ontological commitments that render them not necessary. Wittgenstein has no problem with these criticisms in so far as he also considers statements like (1) to be not necessary. For Wittgenstein, a statement like (1) is not grammatical, but *about* grammar. It is not completely clear from Wittgenstein's writings whether statements like (5) are also grammatical. However, the issue is minor. There is a unique grammatical rule expressed in (1) and (5) and displayed in (4) and the transition from (2) to (3). This rule is the real grammatical proposition.

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11. This nomenclature originates in the work of Zeno Vendler's *Linguistics* in *Philosophy* (Ithaca: Cornell University Press, 1967) 147-171. According to Vendler, a grammatical statement is *external* if it mentions a word, morpheme or any other linguistic entity, and says something about its use. Otherwise, it is *internal*. Consider some examples. Consider the statement ''Spanish' is spelled with a capital 'S'.' This grammatical statement is external, because it mentions the word 'Spanish' and it says something about its use: that it is spelled with a capital 'S'. Now look at one example of internal, grammatical statements: 'Names of languages are capitalized'. This statement does not mention any words, but still expresses a grammatical rule in grammatical vocabulary. Other examples of external, grammatical statements are: ''Dog' is a noun' and 'the gerund of 'walk' is 'walking''. Examples of internal, grammatical statements are: 'Number words function as adjectives' and 'Noun and adjective must agree in gender'. In Chomskian grammars, this distinction corresponds to terminal (external) rules, and non-terminal (internal) rules (of transformation and formation). In the formal reconstruction presented in chapter 4, grammatical propositions that express relations within expressions or between expressions and categories are *internal*, while propositions that express relationships among categories are *external.*

12. This convention is present in the work of W. E. Kennick, "Philosophy as Grammar" in G. E. Anscombe & M. Lazerowitz: *Ludwig Wittgenstein, Philosophy and Language* (Bristol: Thoemmes Press, 1996) and Morris Lazerowitz, "Necessity and Language" in M. Lazerowitz and Alice Ambrose eds. *Essays in the unknown Wittgenstein* (Buffalo: Prometheus Books, 1984). However, it does not completely match with Vendler's notion of 'external grammatical' statement. For Kennick and Lazerowitz, a statement is explicitly grammatical if its vocabulary is grammatical, and implicitly grammatical if it does not include grammatical terms, "but still expresses a rule, convention, or decision about verbal usage."[Lazerowitz. Ibid 142].

Consider, now a mathematical transition, for example, the addition of two numerals under 100. The rule that governs such calculation can be expressed the following way:

(6) The addition of two numerals a∞b and c∞d is the string R(((R(b + d) + a) + c)∞ (((R(b + d) + a) + c)∞(b + d), where R(n) = ‘1’ if n ≥ 10 and R(n) is the empty string otherwise.

This rule applies to the addition of 27 to 34. This calculation may be represented in the following way:

(7)

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  1
  27
_+34__
  61
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This calculation is also expressed in the form of an equation as:

(8) 27 + 34 = 61

Mathematics also features the double nature of grammatical application presented above for the case of natural language grammar. For Wittgenstein, grammatical rules may be applied both in the formation (as in 8) and transformation (as in 7) of expressions.[13]

The application of this rule to numerals ‘27’ and ‘34’ may also be expressed in the following statement:

(9) The result of adding 27 to 34 is 71.

Since the rule governing the calculation is grammatical statements (6), (8) (9), and display (7) are also grammatical. However, just like in the case of (1) to (5), they are all grammatical in different senses. Their relation to the mathematical rule is different.

(7), (8) and (9) all express the same calculation. Yet, when people think about mathematical statements, expressions like (8) most typically come to mind. For Wittgenstein, however, the calculation is displayed in (7) as well as in (8). A mayor difference is that

13. The transformed expressions need not be full statements in the case of internal *Anwendung.*

strings like (8) have the further disadvantage of looking too similar to natural language statements. It is customary to call (8) a mathematical *statement*. However, it is important not to think that, as a statement, it must be about something. Furthermore, it is also important not to infer that its about the addition, either as a calculation or as a mathematical operation. The relation between calculation and mathematical statement is not one of aboutness, but of trace. Statement (8), just like display (7), is the trace left by the calculation. Statement (9), in contrast, expresses this same calculation externally. Unlike (7) and (8), (9) is not a trace of the calculation. It expresses a genuine proposition. This proposition is not the calculation itself. It is *about* the calculation. Even if it does not include explicit mention of numerals, it still lacks the autonomy of belonging to the calculus as (7) and (8) do. In that sense, it is similar to (6). Both (6) and (9) are external mathematical statements. Their meaning is not an autonomous mathematical proposition, but a description of it.

The common criticisms to the grammatical nature of mathematical propositions are dissolved by paying closer attention to the analogy between statements (1) to (5) and (6) to (9). In mathematics, as in ordinary natural language grammar, it is very important to distinguish between grammatical statements and statements *about* grammar. Mathematical statements like (8) and (7) are grammatical, yet they are not about grammar. In strict sense, they are not about anything.

From (7) to (9), there is a unique calculation and, in consequence, a unique mathematical proposition. It is displayed in (7) and (8), but described in (9). Questions about the truth or necessity of mathematical propositions commonly stem from misguided analogies between mathematical statements and descriptive ones. These analogies conceal important differences between the descriptive and mathematical propositions behind the statements. Most of all, they hide the important difference between displaying a rule by following it and describing it.

C. Grammatical Propositions as Rules

As the second appendix to the *Philosophical Remarks*, the editors printed Friedrich Waisman's record of a conversation at Schlick's house, on December 30, 1930, where Wittgenstein drew an analogy between the necessity of a chess-proposition like 'I can force mate in 8' and that of an arithmetical equation. The analogy is based on the simple fact that, besides languages and calculi, other rule-governed practices have produced a specific vocabulary for expressing their rules. In chess, for example, expressions like 'pawn', 'opposite piece', etc. are part of chess vocabulary. Knowing the rules of chess does not only involve learning the permissible moves for each piece, the winning positions, etc. It also requires learning the names of the pieces: what is a check-mate and the like. In other words, knowing chess involves learning the *vocabulary* of the game. This vocabulary is given by the game's 'constitutive rules'.[14] In strict sense, these rules do not say anything about how to play the game, but assist on the understanding of the other rules. They define the meaning of terms within the game.[15]

Consider now, a statement about chess in chess vocabulary. For example: "No two pawns of the same color can be in the same column without having captured an opposite piece." This statement expresses a necessary truth, precisely because it uses chess vocabulary ('pawn', 'capture', etc.). Any acceptable interpretation of this statement must comply with its terms' meanings. The constitutive rules of chess determine these meanings. Hence, any possible interpretation of the statement must already accept those rules. In the conceptual framework that the constitutive rules create, the statement in question expresses a

14. Do not mistake the constitutive rules of chess through which one learns its vocabulary with those ostensive statements assigning pieces' roles to different material objects. An ostensive statement indicating which object, for example, will be a pawn and which one a bishop is not a rule of the game of chess.

15. Describing the game of chess without using these words or some equivalent may be possible, but it would be extremely complicated and artificial.

necessary truth. Since the statement is true according to those rules, its truth is necessary. ‘No two pawns of the same color can be in the same column without having captured an opposite piece’ is necessary.

Someone may say that this proposition is not necessary, insofar as its truth depends on the rules of the game existing as they do, and this is not a necessary fact. The rules of chess could be otherwise, indeed. However, for two pawns of the same color to be in the same column without one capturing an opposite piece is still impossible. For this to happen, the constitutive rules for what is a pawn or what is to capture an opposite piece would have to be different. ‘Pawn’ and ‘to capture an opposite piece’ would have to mean something else. But, in that case, the statement ‘No two pawns of the same color can be in the same column without having captured an opposite piece’ would also mean something different. It would now express a false proposition. However, this false proposition would not be the original proposition. Whatever “no two pawns of the same color can be in the same column without having captured an opposite piece” would mean in this bizarre interpretation of its terms would be false. Yet, it would not be false that no two pawns of the same color can be in the same column without having captured an opposite piece.

If different games had the same vocabulary but different rules, it would be necessary to add a clause to every internal statement indicating in what game to interpret the sentence. If a game different from chess used chess terms but had different rules, it would be necessary to add the clause ‘in the game of chess’ to every chess statement. However, in principle, natural language excludes this possibility. The language of the grammatical statement already suggest the system of rules for interpretation. In Vendler’s words:

> “in saying “One cannot know something false,” I am talking English, so the possibility of interpreting the statement according to the rules of some other language does not arise. To say things like “having a mistress was respectable in Old English but not in current English” is to make a bad joke.

> . . To conclude, a statement such as "One cannot know something false" is not true in English or for English; it is absolutely and categorically true."[16]

The truth of grammatical statements depends on language the same way every other statement does. Every statement is true or false, according to the interpretation rules of the language. Grammatical statements are no different. The only difference is that they determine themselves the rules of their interpretation. This makes them true not only in the language of which they are rules, but in any language "provided they are well translated."[17]

D. Grammar and Vocabulary

The necessity of a chess rule like 'No two pawns of the same color can be in the same column without having captured an opposite piece' does not require that those particular spatial and temporal objects called 'pawns' exist. In this sense, it does not require pawns existing. However, in another sense, it requires the existence of pawns, indeed. If pawns did not exist in chess, that is, if the game of chess were played without pawns, the proposition would be nonsense. In this sense, the proposition requires the existence of pawns to be meaningful. There is no contradiction here, just an equivocation in the understanding of pawns and their existence. In chess, 'pawns' refers both to a kind of piece in the game, and to those material objects playing their role in particular chess matches. The rules of chess determine the pawns' essence in the first sense. However, they are indifferent to pawns in the second sense. In consequence, they require the existence of pawns as pieces defined in the game, but not as material objects. A chess rule does not refer to pawns as spatio-temporal entities existing outside the game. The rule refers to pawns as *pieces* in the game.

16. (Vendler 1967, 24)

17. Ibid. 26

The rules of the game completely define its pieces. 'Being a pawn' is playing a certain role in the game a chess.[18]

> Es ist übrigens sehr wichtig, daß ich den Holzklötzchen auch nicht ansehen kann, ob sie Bauer, Läufer, Turm etc. sind. Ich kann nicht sagen: das ist ein Bauer *und* für diese Figur gelten die und die Spielregeln. Sondern die Spielregeln *bestimmen* erst diese Figur: der Bauer *ist* die Summe der Regeln, nach welchen er bewegt wird (auch das Feld ist eine Figur), so wie in der Sprache die Regeln der Sprache das Logische im Wort bestimmen.[PR Appendix II, p. 315]

> Besides, it is highly important that I can't tell from looking at the pieces of wood whether they are pawns, bishops, rooks, etc. I can't say: that is a pawn *and* such and such rules hold for this piece. No, it is the rules alone which *define* this piece: a pawn *is* the sum of the rules for its moves (a square is a piece too), just as in the case of language the rules define the logic of a word. [PR Appendix II, p. 328]

Similarly, the necessity of a grammatical statement like 'adjectives cannot modify verbs' is not contingent on the material existence of adjectives and verbs, because it does not refer to them as spatio-temporal objects, ink marks on paper, but as grammatical categories. Wittgenstein calls grammatical statements 'internal descriptions', because they address grammatical categories. They are not about objects. The aforementioned statement is not about any objects called 'adjectives', but about the grammatical category 'adjective'. It states its relationship with the grammatical category 'noun'.

By analogy, saying that a mathematical proposition like 'there is no integer between three and four' relies on the integers three and four existing is also equivocal. Since a mathematical statement like '3 + 4 = 7' says that the correct result of adding three to four is seven, its truth does not depend on any spatio-temporal objects or events. As an internal, grammatical statement, it does not refer to additions as spatio-temporal events, but as calculations defined within the mathematical system. It is not contingent on particular instances of numerals '3' and '4' existing either. It does not refer to them. It refers to the

18. The rules determine its essence. In consequence, whatever they say is essential.

numbers three and four. These numbers are the numerals' grammatical categories in the calculus.

Saying that the proposition 'No two pawns of the same color can be in the same column without having captured an opposite piece' is about the word 'pawn' in English is as absurd as saying that an arithmetical equation like '3 + 4 = 7' is about the English word 'seven'. The first proposition is about the pawn piece the sum of chess rules define. Similarly, the arithmetical equation is about the number seven the arithmetical rules define. Michael Young writes,

> In calculating we do deal with a particular collection of characters or marks, say those that we have written on a piece of paper, but it should be clear that we do not deal with them as perceptual objects in their own right, attributing to them whatever properties they might happen to exhibit. If one '6' happens to be larger than another instance, or to have a different color or shape, we recognize that this is quite irrelevant. We treat the characters that we intuit merely as instances of the Arabic numerals, ignoring everything else about them.[19]

The way a genuine proposition like 'the current king of France is bald' relies on the existence of the current king of France (or a chess statement requires the existence of chess pieces) is significantly different from the way an arithmetic proposition relies on numbers existing. Without a current French king, any external description of the current king of France would be nonsensical. Without chess, the statement 'No two pawns of the same color can be in the same column without having captured an opposite piece' would not be false, but absurd. Without the existence of numbers 3, 4 and 7, '3 + 4 = 7' would be just a senseless string of marks on paper. However, the similarities stop here. Unlike genuine propositions or chess rules, mathematical propositions are autonomous. Since the calculus is its own internal *Anwendung*, mathematical statements do not describe mathematical rules, they are themselves mathematical rules.

19. (Young 1982, 25)

Mathematical propositions are syntactically necessary, because they cannot belong to a calculus and be false. They require no more than their own calculus. Without numbers, arithmetical equations would not exist. If 7 were not a number, $3 + 4 = 7$ would not be an arithmetical equation. Nevertheless, if the arithmetical equation exists, the numbers and operations involved in it exist too. The arithmetical equation $3 + 4 = 7$ is grammatically necessary, because its existence in the calculus guarantees the existence of its terms. The existence of the equation $3 + 4 = 7$ in the calculus guarantees that 3, 4 and 7 also exist.

The equation $3 + 4 = 7$ guarantees more than 3, 4 and 7 existing in the calculus. It also guarantees that adding three to four is seven. Otherwise, the equation would not belong to the calculus, either. It would not be an arithmetic proposition. If adding three to four was not seven, '$3 + 4 = 7$' would not be a false arithmetical proposition. It would not be an arithmetic proposition at all. The connection between a genuine proposition and whatever it is about differs radically from the connection between a mathematical proposition and a calculation. Genuine propositions describe possible states of affairs. Mathematical propositions do not describe calculations. They are themselves calculations. The truth of a genuine proposition like 'the cat is on the mat' requires the cat being on the mat. A mathematical proposition does not require anything that it does not construct for itself.

III. Wittgenstein's Syntactic Necessity as Analyticity

A. Brief Historical Background

According to the middle Wittgenstein, internal descriptions ascribe essential properties to objects, while external descriptions ascribe accidental properties.[20] A description is internal if the concept in the subject includes or implies the concept in the predicate. This

20. PR §94.

characterization of internal descriptions is very close to that of analytical statements. Since Wittgenstein also includes mathematical statements among internal descriptions, this commits him to believe that mathematical statements are analytic.

Before the linguistic turn in philosophy at the end of the nineteenth century, Locke had already distinguished two kinds of analytic propositions. In *An Essay concerning Human Understanding* [pp. 306, 308], he distinguished between ‘trifling’ and ‘predicative’ propositions. Trifling propositions have the form ‘$a = a$’, in which “we affirm the said term of itself.” In predicative propositions, “a part of the complex idea is predicated of the name of the whole.” For Locke, mathematical propositions are not analytic in either of these senses. After Locke, Kant added a new account of analyticity to Locke’s notion of trifling proposition. For Kant, an analytic judgement is (*i)* one whose subject concept contains its predicate concept, or (*ii*) one whose negation is a logical contradiction. By offering these two different accounts, Kant laid the foundations for what became the two main doctrines of analyticity in modern western philosophy.[21] For Kant, a judgement is analytic if the subject’s concept contains the predicate’s concept. However, he allows for two possible interpretations of this ‘containment’, what Jerrold J. Katz in *The New Intentsionalism* calls ‘logical-containment’ and ‘concept-containment’.[22] Kant’s notion of ‘analytic’ fused these two notions, as they remained until Frege separated them. For Frege, Kant’s account of analyticity in terms of conceptual containment was a psychologistic error. In *The Foundations of Arithmetic* §3, Frege defines analyticity as ‘being a consequence of logical laws plus definitions without scientific assumptions’. Wittgenstein’s account of logical necessity in the *Tractatus* follows Frege away from the conceptual path and into logicism. This path leads from Wittgenstein directly into the Quine/Carnap controversy. At the end of

21. They could be called the ‘logicist’ and the ‘idealist’ doctrines.

22. Jerrold J. Katz, *Mind*, New Series, vol. 101, issue 404 (Oct., 1992), 691.

his 1944 article on ‘Russell’s mathematical logic’ Kurt Gödel distinguishes two senses of ‘analyticity’.

> As to this problem [if (and in which sense) mathematical axioms can be considered analytic], it is to be remarked that analyticity may be understood in two senses. First, it may have the purely formal sense that the terms occurring can be defined (wither explicitly or by rules for eliminating them from sentences containing them) in such a way that the axioms and theorems become special cases of the law of identity and disprovable propositions become negations of this law. . .
>
> In a second sense a proposition is called analytic if it holds “owing to the meaning of the concept occurring in it”, where this meaning may perhaps be undefinable (i.e., irreducible to anything more fundamental). [Note 47. The two significations of the term ‘analytic’ might perhaps be distinguished as tautological an analytic.][23] According to Carnap, Wittgenstein’s *Tractatus* endorsed the view of mathematical propositions as analytic in the tautologous sense. However, by the beginning of the thirties, Wittgenstein’s view on the analyticity of mathematics had evolved from a purely formal notion into Gödel’s second sense.
>
> After the *Tractatus*, considerations about color and the nature of space changed Wittgenstein’s mind about the logicist’s path. In the *Tractatus*, he had mantained that “there is only *logical* necessity” [6.375]. However, by the late twenties, he could hardly see how the *Tractatus*’ logical necessity could account for the necessity of such propositions as ‘The blue spot is not red at the same time’. In the early thirties, the notion of grammatical necessity had become a substitute for that of logical necessity in the *Tractatus*.

B. Carnap

The debate between Carnap and Quine – and, by extension, Tarski, Gödel, Dummett, Putnam, et. al. – concentrates on mathematics as part of the formal syntax of language. Because Carnap asserted that his thesis of mathematics as syntax sprang from Wittgenstein, taking a stance regarding this debate is critical. Clarifying whether or not he held a view like the one Carnap championed is vital, as is defending Wittgenstein against Quine’s criticisms.

Despite their mutual personal dislike,[24] Carnap always recognized Wittgenstein’s

23. (Gödel: 1986, 139)

24. On March 27, 1998, as part of an electronic exchange in the *Foundations of Mathematics* mailing list,

influence on this and other philosophical matters. In his “Intellectual Auto-biography,”Carnap states that “Wittgenstein was perhaps the philosopher who, besides Russell and Frege, had the greatest influence on my thinking.”[25] From Carnap’s own appraisal, the sources of this influence were triple: (*i*) careful and intense reading of the *Tractatus* by the Vienna Circle, (*ii*) personal contact between Carnap and Wittgenstein from the Summer of 1927 to the beginning of 1929, and (*iii*) “Waismann’s systematic expositions of certain conceptions of Wittgenstein’s on basis of his talks with him.”[26]

According to Carnap,

> “The most important insight I gained from his [Wittgenstein’s] work was the conception that the truth of logical statements is based only on their logical structure and the meaning of terms. Logical statements are true under all conceivable circumstances; thus their truth is independent of the contingent facts of the world. On the other hand, it follows that these statements do not say anything about the world and thus have no factual content.[27]

From Wittgenstein, Carnap received the idea that logical truths are tautologies. In the *Tractatus*, Wittgenstein unsuccessfully argued for the tautologous nature of logical truth for the first time in the history of logicism.

However, the issue of logical truth is the source of both the main agreement and most important divergence between Carnap and Wittgenstein. According to Michael Friedman,

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Neil Tennant wrote: “It is worth recording here that Carnap---perhaps the most influential member of the V.C. as far as philosophy of mathematics and foundations are concerned---and Wittgenstein did not hold one another in very high regard, despite Carnap's always polite and forbearing published comments on the influence of Wittgenstein.” Tennant offers evidence documented from his study of the Carnap archives. Unlike the apparently personal attacks, Tennant documents the priority debate between Wittgenstein and Carnap in his paper “The Life and Work of the Early Carnap,” in N. Rescher ed., *Scientific Inquiry in Philosophical Perspective*, University Press of America, 1987, pp.261-280. To: parsons2@fas.harvard.edu Subject: “Re: FOM: Wittgenstein: two clarifications.” From: Neil Tennant <neilt@mercutio.cohums.ohio-state.edu>. Date: Fri, 27 Mar 1998 13:40:26 -0500 (EST). Sender: owner-fom@math.psu.edu.

25. (Schilpp 1963, 46)

26. Ibid. 28

27. Ibid. 25

> This conception of the tautologous character of logical and mathematical truth represents Carnap, the most important point of agreement between his philosophy and that of the *Tractatus*. But there is also an equally important point of fundamental disagreement. Whereas the *Tractatus* associates its distinctive conception of logical truth with a radical division between what can be said and what can only be shown but not said a division according to which logic itself is not properly an object of theoretical science at all – Carnap associates his conception of logical truth with the idea that logical analysis, what he calls "logical syntax," is a theoretical science in the strictest possible sense.[28]

In terms of the middle Wittgenstein, the main point of divergence between Carnap and Wittgenstein was the autonomous character of mathematics and grammar. Under the influence of Frege and Russell, Carnap was always convinced of "the philosophical relevance of constructed language systems."[29] During his years in the Vienna Circle, Otto Neurath nurtured Carnap's idea that a descriptive science of the structure of language – what would become the "Logical Syntax of Language" – was possible. Finally, Carnap's study of Hilbert and his continuous talks with Tarski and Gödel convinced him of the philosophical power of meta-mathematics. By the time he had developed his theory of logical syntax, virtually all connection with Wittgenstein's notion of tautology and analyticity seemed lost.[30] Most strikingly, Carnap's logical syntax of language, unlike Wittgenstein's grammar, had lost its autonomy.

Carnap conceived of philosophy as a descriptive, scientific enterprise geared towards formulating the logic of science in a precise meta-language.[31] Instead of an indescribable, but displayable grammar, Carnap expresses his logical syntax in its own object language. Carnap uses Gödel's arithmetization method to embed the syntactic meta-language in the

28. Michael Friedman, "Carnap and Wittgenstein's *Tractatus*" (Tait 1997, 20)

29. (Schilpp 1963, 28)

30. (Friedman 1997, 23)

31. Rudolf Carnap, *The Logical Syntax of Language* (New York: Harcourt, 1937) §73.

object language (provided that the object language includes elementary arithmetic), allowing it to express its own syntax. However, it immediately follows from Gödel's work that, for a language containing classical arithmetic, 'truth' is a non-arithmetical predicate and thus, undefinable in the language itself. Carnap understood this and, hence, qualified his remarks on this method in *Logical Syntax*. Commenting on the Wittgenstein / Carnap connection, Michael Friedman interprets this as a point in favor of Wittgenstein's autonomous grammar over Carnap's logical syntax.[32]

The failure of Carnap's attempt to syntactically define analyticity is a point in favor of the autonomy of mathematics. Carnap followed Wittgenstein's search for mathematics in the syntax of language. Natural language grammar contains embedded mathematical calculi. However, Carnap was wrong in thinking that mathematics describes this external *Anwendung* in a meta-language. Mathematics is autonomous. Every calculus is its own internal *Anwendung.* This internal *Anwendung* does not require a metamathematical formulation. The calculus is sufficient.

It is possible to describe a calculus' external *Anwendung* in a meta-language. Chapter 3 is an example of this. However, this description is *not* the calculus itself. Describing a syntax is substantially different from calculating. Unlike calculation, description is not autonomous. The truth of a descriptive proposition point outside the description itself. Calculation determines the correctness of its own propositions. The mere

32. "Carnap, characteristically, has transformed an originally philosophical point into a purely technical question. – in this case, the technical question of what formal theories can or cannot be embedded in a given object language. Considered purely as a technical question, however, the situation turns out to be far more complicated than it initially appears. . . For it turns out, again as a consequence of Gödel's researches, that it is as a matter of fact not possible in most cases of interest to express the logical syntax of a language in Carnap's sense in the language itself. . . Thus, the logical syntax in Carnap's sense for a language for classical mathematics can only be expressed in a distinct and essentially richer metalanguage; the logical syntax for this metalanguage can itself only be expressed in a distinct and essentially richer meta-metalanguage; and so on. . . Does this same situation does not represent the kernel of truth – from Carnap's point of view, of course – in Wittgenstein's doctrine of the inexpressibility of logical syntax?" (Friedman 1997, 35-36)

description of a calculus' external *Anwendung* cannot fully determine the correctness or incorrectness of its propositions. Gödel showed that Carnap's attempt failed technically. Wittgenstein showed that the project was also philosophically inadequate.

C. Quine

1. Two Dogmas and the Analytic Nature of Grammar

> The linguistic doctrine of logical truth is sometimes expressed by saying that logical truths are true by linguistic convention.
> Quine 1963, 391

The analytic/synthetic distinction has a long history in modern philosophy. According to Quine's "Two Dogmas of Empiricism", the writings of Leibniz, Hume and Kant foreshadow the contemporary distinction. However, both Hume's "relations of ideas" and Leibniz's "truths of reason" are quasi-psychological notions. It was Kant who first inserted language at the core of the philosophical characterization of analyticity. The idea of 'truths independent of fact' precedes Kant. Nonetheless, starting with him, these truths became also 'true by virtue of meaning'. The current notion of 'analyticty' originates in Kant. After the seminal work of Frege, analyticity secured a central place in contemporary philosophy of logic and mathematics. The discussion of analyticity in this century has grown largely from his conception. Nevertheless, Quine offered the principal arguments against the analytic/synthetic distinction, not in response to Frege, but in response to Carnap's *The Logical Syntax of Language.*" Those arguments are so convincing that even today a large number of philosophers and mathematicians consider some of the points made in these seminal writings settled matters. For example, Paul Artin Boghossian, starts his 1996 article 'Analyticity Reconsidered' with the following remarks:

> This is what many philosophers believe today about the analytic/synthetic distinction: In his classic early writings on analyticity — in particular, in "Truth by Convention," "Two Dogmas of Empiricism," and "Carnap and

> Logical Truth" — Quine showed that there can be no distinction between sentences that are true purely by virtue of their meaning and those that are not. In so doing, Quine devastated the philosophical programs that depend on he notion of analyticity — specifically, the linguistic theory of necessary truth . . . Now, I do not know precisely how many philosophers believe all of the above, but I think it would be fair to say that it is the prevailing view.[33]

Quine's strategy against the analytic/synthetic distinction is stunningly novel and elegant. It targets its putative linguistic dimension through the syntax/semantics distinction. For Quine, if some propositions are true in virtue of linguistic conventions, then either their syntax or their semantics determines their truth. In "Two Dogmas," he distinguishes between '*logically true*' (syntactic) and others (semantic) analytic statements.[34] According to Quine, both the proof theoretical and model theoretical approaches to necessity can only account for analytic statements of the first kind. For the rest of the article, Quine attacks different attempts – mostly Carnap's – at reducing analytic sentences of the second class to those of the first class. According to Quine, Carnap's account of analyticty is unsuitable, because it tries to reduce semantics to syntax. For Quine, 'analytic' is an irreducible semantic notion. He finds no non-circular, suitable, semantic account of analyticity. For Quine, the usual attempts at semantically defining analyticity are circular, because they require a previous semantic understanding of analyticity.

Wittgenstein's account of analyticity is not semantical, but syntactic. However, it does not correspond fully to Quine's notion of logical truth. Quine's definition of logical truths reformulates Yehoshua Bar-Hillel's reconstruction of Bolzano's definition of analytic proposition.[35]

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33. *Nous* 1996.

34. W. V. O. Quine, "Two Dogmas of Empiricism" *The Philosophical Review* 60, no. 1 (January 1951) 23.

35. "If we suppose a prior invertory of *logical* particles, comrpising 'no', 'un-', 'not', 'if', 'then', 'and', etc., then in general a logical truth is a statement which is true and remains true under all reinterpretations of its components other than the logical particles." Ibid. 23

> First, we suppose indicated, by enumeration if not otherwise, what words are to be called logical words; typical ones are 'or', 'not', 'if', 'then', 'and', 'all', 'every', 'inly', 'some'. The logical truths, then, are those true sentences which involve only logical words *essentially*. What this means is that any other words, though they may also occur in a logical truth (as witness 'Brutus', 'kill', and 'Caesar' in 'Brutus killed or did not kill Ceasar'), can be varied at will without engendering falsity.[36]

As a matter of fact, Wittgenstein's grammatical method is indeed very similar to one of the attempts at defining analyticity syntactically discussed in "Two Dogmas". In section III, Quine discusses the account of analyticity, according to which (*i*) "any analytic statement could be turned into a logical truth by putting synonyms for synonyms"[37] and (*ii*), (cognitive) synonymy[38] is "interchangeability *salva veritate* everywhere except within words."[39] According to him, this latter account is flawed, because interchangeability *salva veritate* does not capture cognitive synonymy, but only coextensionality. In consequence, not only analytic truths, but also synthetic truths may be transformed into logical truths through *salva veritate* substitution. For example, since the current president of Mexico in August 2000 is Ernesto Zedillo, the singular terms 'current president of Mexico in August 2000' and 'Ernesto Zedillo' are interchangeable *salva veritate*. In consequence, substituting 'current president of Mexico in August 2000' for 'Ernesto Zedillo" in the logical truth 'The current president of Mexico in August 2000 is the current president of Mexico in August 2000' results in the synthetic truth 'The current president of Mexico in August 2000 is Ernesto Zedillo'. An endorser of this account may object that distinguishing between 'current president of Mexico in August 2000' and 'Ernesto Zedillo' remains possible. The terms cannot substitute for each other in a sentence like 'Necessarily the

36. (Quine 1963, 387)

37. (Quine 1951, 28)

38. Quine distinguishes cognitive analyticity from "synonymy in the sense of complete identity in psychological associations or poetic quality." [p. 28]

39. (Quine 1951, 28)

current president of Mexico in August 2000 is the current president of Mexico in August 2000', because 'Necessarily Ernesto Zedillo is the current president of Mexico in August 2000' is false. However, Quine retorts, this objection begs the question.

> The above argument supposes we are working with a language rich enough to contain the adverb "necessarily', this adverb being so construed as to yield truth when and only when applied to an anlytic statement. but can we condone a language which contains such an adverb? Does the adverb really make sense? To suppose that it does is to suppose that we have already made satisfactory sense of 'analytic'. Then what are we so hard at work on right now?[40]

It is clear that Wittgenstein's grammatical method is very similar to that of Section III in "Two Dogmas". However, they are also significantly different, and these differences are strong enough to elude Quine's criticisms. First of all, Wittgenstein's interchangeability criterion is not *salva veritate*, but *salva grammaticality*. Second, it is not an attempt at defining general synonymy, but grammatical synonymy. In other words, it applies only to grammatical terms, not to all terms in general. Hence, it does not attempt to reduce genuine semantics to syntax – certainly a doomed enterprise. It attempts to give a synonymy criteria for those words whose grammar entirely determines their meaning.

Wittgenstein's distinction between grammatical and genuine propositions is similar to that between analytic and synthetic statements. However, Wittgenstein's distinction presumes nothing about its empirical nature, while Quine's primary concern is with the empirical dimension of the analytic/synthetic distinction. Wittgenstein's distinction between grammatical and genuine propositions is closer to the current distinction between syntax and semantics. Wittgenstein bases his distinction at the level of propositions on a distinction at the level of concepts and objects (as shown in chapter 2). For Wittgenstein, grammatical terms are those whose grammar entirely determines their meaning. Since grammatical

40. (Quine 1951, 29)

concepts lack intensionality, co-extensionality offers suitable criteria for synonymy among grammatical terms.

Indeed, Wittgenstein never maintained that grammar fully determined the meaning of all terms. However, he argued that it did for those he called ‘grammatical’. In Wittgenstein’s grammatical method, grammatical concepts are grammatical categories, given by linguistic contexts. Two terms are grammatically equivalent if they are interchangeable *salva grammaticality* in all contexts. If the terms are grammatical, they are also synonymous. They have the same grammatical category as their meaning.

At the level of statements, a statement is grammatical if its concepts are grammatical concepts. In consequence, its grammar completely determines its ‘meaning’ and ‘truth’. In contrast, grammar cannot fully determine the truth of genuine propositions, but only their possibility. If a non-grammatical statement is well-formed, its meaning is a genuine proposition. It expresses a possible state of affairs. Modality is already built into the grammar of the language. In consequence, Wittgenstein’s grammatical account does not require a previous understanding of synonymy and, hence, is not circular in Quine’s sense.

2. Convention and Justification

> But still there was no truth by convention,
> because there was no truth.
> Quine 1963, 392

The breadth of Quine’s arguments in “Truth by Convention” focuses on the foundational role of linguistic conventions. In consequence, it is mostly irrelevant for Wittgenstein’s grammatical project. Clearly, Wittgenstein found such a foundational enterprise absurd. His philosophy of mathematics during the middle period is not a conventionalism in that sense.

The target of Quine’s anti-conventionalist arguments is linguistic conventions’ inability to found mathematics or calculus. In other words, in “Truth by Convention”, Quine questions linguistic conventions' capacity to justify mathematical or logical truths. However, Wittgenstein's grammatical account of mathematics is not a foundational enterprise. In Wittgenstein's account, grammatical rules certainly have no justificatory power. Wittgenstein most likely would sympathize with Quine's efforts to demonstrate the impossibility of justifying logical and mathematical truths by inferring them from syntactic conventions.

> . . . the difficulty is that if logic is to proceed *mediately* from conventions, logic is needed for inferring logic from the conventions.[41]

Dummett reiterates this criticism when he says, in his "Wittgenstein on Necessity":

> The moderate conventionalist view was never a solution to the problem of logical necessity at all, because, by invoking the notion of consequence, it appealed to what it ought to have been explaining: that is why it appears to call for a metanecessity beyond the necessity it purported to account for. The conventionalists were led astray by the example of the founders of modern logic into concentrating on the notion of logical or analytic *truth*, whereas precisely what they needed to fasten on was that of deductive *consequence*. . .[42]

Wittgenstein would agree with Quine and Dummett that logical truths and linguistic conventions do not entail each other logically. If conventions logically entailed logical truths, justifying this relation would itself require logic. 'Logical entailment' and 'justification' are concepts that do not apply to grammatical propositions, at least not in the same sense as they apply to genuine propositions.

If 'to justify *p'* means to demonstrate the truth of *p*, then justification applies only to genuine propositions. Correct calculations are also called mathematical truths, but mathematical truth is not a sub-species of 'truth' in general. For Quine, "We may mark out the intended scope of the term 'logical truth', within that of the broader term 'truth'."[43] How-

41. W. V. O. Quine, *Mathematical Logic* (Cambridge: Harvard University Press, 1951) 97.

42. Michael Dummett, *The Seas of Language* (Oxford: Clarendon Press, 1993) 460.

43. Quine [1963] p. 386.

ever, since the *Tractatus*, the scopes of 'truth' and 'logical truth' do not overlap for Wittgenstein. In Ramsey's words, "It is important to see that tautologies are not simply true propositions, though for many purposes they can be treated as true propositions."[44] Ramsey presented Wittgenstein's position very clearly in his 'Foundations of Mathematics' where he wrote:

> The assimilation of tautologies and contradictions with true and false propositions respectively results from the fact that tautologies and contradictions can be taken as arguments to truth-functions just like ordinary propositions, and for determining the truth or falsity of the truth-function, tautologies and contradictions among its arguments must be counted as true and false respectively. Thus, if '*t*' be a tautology, '*c*' a contradiction, '*t* and *p*', 'If *t*, then *p*', '*c* or *p*' are the same as '*p*', and '*t* or *p*', 'if *c*, then *p*' are tautologies.[45]

For Wittgenstein, 'truth' means something different when applied to tautologies than it does when applied to genuine propositions. In the logical calculus of propositions, being true is having 'truth' as truth value. In the truly semantic case, being true means that the proposition is the case. "For what does a proposition's '*being true*' mean? '*p*' *is true* = *p*. (That is the answer.)"[46] In the case of tautologies and contradictions, nothing could or could not be the case. In consequence, saying that they are true (or false for that matter) in the same sense as true genuine propositions makes no sense. The predicate 'true', defined for genuine propositions, does not apply to tautologies or contradictions.

Mathematics is pure calculus, and every calculus is a rule-governed practice. In this respect, calculi are more like chess than like natural science. Asking for the justification of a mathematical truth is like asking for the justification of the truth of chess rules. Both are nonsense. It makes sense to justify 'that *p*', but not to justify '**to** *p*'. Unless justification

44. Frank P. Ramsey, 'The Foundations of Mathematics (1925)' in *Philosophical Papers* (Cambridge: Cambridge University Press, 1990) 173.

45. Ibid. 174.

46. RFM Pt. I appendix I, §6

means something different when applied to rules and practices than to genuine propositions. For Carnap, a calculus is as ‘justified’ as its application. Carnap’s conventionalism is also a pragmatism. Application justifies calculation. Wittgenstein offers a different interpretation. For him, following a rule justifies it. A rule is justified if it is possible to follow it. This sense of justification does not require metamathematics. Performing the calculation is sufficient. It demonstrates that following the rule is constructively possible. Wittgenstein's grammatical necessity is the necessity of calculations, not of propositions.[47]

Finally, Wittgenstein is not a conventionalist in Dummett’s sense either. According to Dummett, Wittgenstein is a *radical* conventionalist, because he grounds mathematical necessity on the *decision* of not questioning mathematical truth. However, for the middle Wittgenstein, deciding whether or not to question mathematical propositions is absurd. Questioning grammatical propositions does not make sense. Accordingly, the mere notion of such a decision is nonsensical. Mathematical propositions are not the kind of things it makes sense to question. Hence, mathematics contains no decisions and, in consequence, no radical conventions, either.[48]

IV. Conclusion: Wittgenstein’s Own Account of Analyticity

Wittgenstein’s grammatical account of the analyticity of internal descriptions in general, and mathematical propositions in particular, differs from Carnap and most recent accounts of analyticity, because it is not metaphysical or epistemological, but logical. In his 1996 article ‘Analyticity Reconsidered’, Paul Boghossian distinguishes between two different notions of analyticity: a metaphysical and an epistemological one.

47. His interest is precisely what Dummett calls necessary consequence: what necessarily *follows* according to a rule.

48. Do not confuse convention with stipulation. Grammatical rules may be conventions, but they are certainly not stipulations.

> Here, it would seem, is one way: *If mere grasp of **S**'s meaning by **T** sufficed for **T**'s being justified in holding **S** true. . .* On this understanding, then, 'analyticicty' is an overtly *epistemological* notion: a statement is 'true by virtue of its meaning' provided that grasp of its meaning alone suffices for justified belief in its truth.
>
> Another, far more *metaphysical* reading of the phrase 'true by virtue of its meaning' is also available, however, according to which a statement is analytic provided that, in some appropriate sense, it *owes its truth value completely to its meaning*, and not at all to 'the facts'.

Wittgenstein's analyticity is neither metaphysical nor epistemological. Wittgenstein agrees with Kant that separating analyticity from aprioricity is important. 'Ananlytiticty' is a logical notion, while 'apriori' is epistemological. However, Wittgenstein understands analyticity not as 'true by virtue of meaning', but 'true by virtue of grammar.'[49] Grammatical analyticity is not a semantic notion, but a logical one. Wittgenstein's account says that a statement **S** is analytic if and only if the mere inclusion of **S** in the language suffices for its truth. The term 'inclusion' in this characterization misleads, since the proposition does not exist outside **S**. Accordingly, the mere existence of **S** guarantees its truth. A grammatical statement **S** cannot exist and be false.

The truth of a mathematical calculus is not contingent on the existence of genuine objects, but only those mathematical ones it constructs for itself. No calculus requires the

49. However, some scholars consider Wittgenstein's analyticity epistemological. Alberto Coffa ["Carnap, Tarski and the Search for Truth," *Nous* 21, no. 4 (December 1987) : 547-572] interprets Wittgenstein's account of analyticity – from the *Tractatus* to the middle and late periods of his philosophy – as epistemological. Wittgenstein characterizes logical sentences in the *Tractatus* as those "one can recognize [*erkennen*] from the symbol alone that they are true" [6.113] Coffa also recognizes "that this determination is embodied in constructive procedures that allow someone who understands the given langauge to 'recognize' the truth-values in question." [pp. 547, 548] Nevertheless, he does not interpret this procedure as a syntactic/grammatical one, but as an epistemic one. Michael Hymers ["Internal Relations and Analyticity: Wittgenstein and Quine" *Canadian Journal of Philosophy* 26, no. 4 (December 1996) : 591-612] also sustains that Wittgenstein's criteria for recognizing analytic propositions [internal descriptions] remained epistemological from the *Tractatus* to PG. [p. 594] He writes, "Also implicit here [in the *Philosophical Grammar*], is a further revision of the epistemic criterion for internal relations: two concepts, or instruments of language, are internally related if in order to understand one I must also understand the other. . . However, concepts have no existence here, independently of norms and practices. Understanding a concept is, paradigmatically, to be able to use a word correctly, where correctness ammounts to accord with the rules of a calculus." [pp. 596-597]

existence of any spatio-temporal objects or events. For example, arithmetical addition is not contingent on any particular numerals, or additions. A mathematical statement like ‘3 + 4 = 7’ says that the correct result of adding three to four is seven. However it does not refer to any particular numerals or additions. The equation refers to numbers as *roles* in the calculus and to additions as calculations: entities fully defined by the calculus’ rules.

Mathematics is part of the syntax of language. However, mathematics is not describing this syntax in a metalanguage. Describing syntax is substantially different than calculating. A meta-linguistic description of logical syntax, like Carnap’s, is external, while mathematics is autonomous. The mere description of a calculus’ external *Anwendung* cannot fully determine the correctness or incorrectness of its propositions.

Mathematics is pure calculus, and mathematical propositions are calculation rules. ‘Justification’ and ‘truth’ apply to genuine propositions only. They do not apply to rules. Justifying a mathematical truth is as absurd an enterprise as justifying the truth of a chess rule, for example. Grammatical necessity is the necessity of calculations, not of propositions. It requires no further justification. Performing the calculation is enough to guarantee its ‘truth’, because a calculation is autonomous. It cannot exist as a calculation and be false.