From Mathematical Possibility to Existence

The case of Shapiro

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The problem of mathematical knowledge is usually characterized by referring to the classic dilemma suggested by Benacerraf against Platonism, which is to show that the answers to the questions

a) What does the discipline deal with? and

b) How do we have knowledge of it?

are conflicting. The answer to the former, if we suppose that the semantics of mathematical statements is analogous to that of statements in ordinary language, is that mathematics deals with objects such as numbers, polyhedrons, vector spaces, etc., which, according to the science itself, are neither temporal nor spatial, nor indeed part of any causal chain. In contrast to this, Gettier suggested in 1963[1] that for an agent A to have knowledge that P, the factors that make P true should have some causal relationship with the reasons why A believes P. Benacerraf's conclusion is that in order to explain mathematical knowledge, we should renounce Platonism (according to which mathematical objects exist independently of our cognitive abilities) or we should show that the semantics of ordinary language and mathematical language are not similar. Although the causal theory of knowledge has fallen into disuse, Benacerraf's problem has survived. The dilemma is now framed in terms of naturalism: if, as science indicates, man is a creature bounded by space-time, how can he have knowledge of abstract objects?

One might easily imagine that one might find a path to a solution by focusing on the fact that mathematicians do not consider a proposition true until they have proof of it, and that therefore everything resides in how we understand the demonstration. However, there is an underlying difficulty. A proof only shows that a certain proposition is a consequence of the axioms of the theory and the problem then shifts to questions of what logical knowledge is and how we know the axioms.

Three ways of responding to Benacerraf's challenge have had few repercussions and will not be the object of our study. The first is Gödel's, according to which man is not a being bounded by space-time but rather has an ability that allows direct contact with abstract objects, similar to the awareness of the ordinary things that surround us provided by sensory perception. The second is from P. Maddy's early work and supposes that certain mathematical objects (some impure sets) are part of our sensory world. The third is rationalism (for example, that of Katz), that renounces the restriction imposed by naturalism.

Another way of confronting Benacerraf's problem is Quine and Putnam's so-called indispensability argument. According to this, all objects postulated by our best theory of the world exist. The objections to this approach and their main consequences have been amply studied by Colyvan.

In this talk, I shall concentrate on another way of dealing with Benacerraf's problem, one that does not suppose that we have contact with abstract objects either, and which has received less attention, perhaps because it has been employed by authors

[1] (Gettier, 1963)

belonging to diverse schools. It is the principle that the objects characterized by the axioms of a consistent theory exist. Formulated explicitly for the first time by Hilbert in his controversy with Frege[2], this principle underlies the Platonism and fictionalism defended by Balaguer[3], and the structuralism of S. Shapiro[4], among others. If the objects defined by the axioms of a consistent theory exist, then they are, of course, just as the theory tells us that they are. In this way, we would be able to overcome the dilemma mentioned above, so long as we could define or characterize logical consistency in such a way that the knowledge that a theory is consistent does not involve, in its turn, any knowledge of abstract objects. This last question is more general, because it not only concerns those in favour of this principle, but also certain nominalists and fictionalists who reduce mathematical knowledge to logical knowledge.

To the best of my knowledge, Hilbert was the first to adopt FBP explicitly. In a passage from his correspondence with Frege, Hilbert says: 'you write... "Of the truth of axioms, it follows that they do not contradict each other..." I have been saying the exact opposite: if axioms arbitrarily given do not contradict each other, including all their consequences, then they are true and the things defined by the axioms exist. For me, this is the criterion of truth and existence.'[5] In an article written some years later, this principle is ratified and clarified: 'The proof of the consistency of axioms requires no more than a proper modification of the usual methods of deduction. In this proof, I seem to glimpse equally the proof of the existence of the set of real numbers … All the doubts and objections that have been stated in relation to the existence of the set of real numbers and, in general, in relation to the existence of infinite sets appear to be unjustified once we have adopted the approach that we have just described. Following on from that, we do not have to understand the set of real numbers as the whole of possible laws according to which the elements of a fundamental series can advance, but rather, as we have just said, a system of objects whose relationships are determined by the *finite and closed* system of axioms I-IV, and in relation to which no statement will be valid if it cannot be deduced from those axioms by means of a finite number of logical inferences.' We will not stop to see how Hilbert used this principle, nor if he gave arguments in its favour.

One might think that Gödel's completeness theorem provides a basis for Hilbert's assertion since it shows that if $\Gamma \cup \{\alpha\}$ is a set of first order sentences and if α is a logical consequence of Γ, then α can be derived from Γ in the CP. Or, to put it another way, if it is (syntactically) impossible to derive a contradiction from a first order theory T, then T has a (countable) model. However, the theorem cannot be used in favour of FBP because its proof supposes a metaphysical combinatorialism. More specifically, if the theorem must validate FBP, we must suppose that there exists a countable set of objects (signs) and that it has the properties and relationships that we wish to assign to them arbitrarily. That is to say, we have to begin from a previously given ontology.

In 1997, in one of the fundamental texts of mathematical structuralism, Shapiro defends a similar idea to Hilbert's: 'Mathematical objects are tied to structures, and a structure exists if there is a coherent axiomatization of it. A seemingly helpful consequence is that if it is possible for a structure to exists, then it does. Once we are satisfied that an

[2] (Frege, McGuinness, Kaal, 1980). See Frege to Hilbert, 29/12/1899.
[3] (Balaguer, 1998), p. 5.
[4] (Shapiro, 1997), pp. 118 y 105.
[5] (Hilbert to Frege, 29.12.1899), Frege Gottlob (1980), p. 39.

implicit definition is coherent, there is no further question concerning whether it characterizes a structure.'[6]

How does Shapiro understand this principle and what does he argue in its favour? In particular, what should we understand by 'object', 'structure', 'coherent' (or 'possible') and to what type of existence does FBP refer?

Shapiro defends a form of realism in ontology and truth value. That is to say, he argues that mathematical objects exist and that meaningful mathematical statements have an objective truth value. More specifically, he maintains that mathematical statements, in their standard formalization, should be taken literally (*at face value*) and therefore as making reference to the values along which their quantified variables range. Thus interpreted, they are true and the objects to which the theory is ontologically committed exist.[7]

The school of realism that Shapiro defends is structuralism. This school maintains, for example that numbers are not independent of the structure of which they are positions; not only that, but that they lack internal composition: they do not have properties other than those that result from their occupying positions in the structure of natural numbers. Shapiro distinguishes between *system* and *structure*: 'I define a system to be a collection of objects with certain relations… a structure is the abstract form of the system, highlighting the relationships about the objects, and ignoring any features of them that do not affect how they relate to other objects in the system.'[8] For example, the structure of the natural numbers is exemplified by, among other things, the system of von Neumann numerals. We can look at these from two points of view.[9] For example, if we say that the president of the United States will always have more power than the vice-president, we can understand this as referring to a concrete system, that is to say, to the individuals who occupy these respective positions at that time. That is the 'places -are-roles' point of view. Similarly, we can understand it as referring to the structure itself, that is to say, referring to the fact that, according that the laws of that country, the president is invested with more power that the vice-president. That is the 'places -are-objects' point of view. In the 'places-are-roles' point of view, there is a background ontology of the objects that occupy the positions. This ontology can also be given by places in other structures or by places in the same structure. This is why, for Shapiro, the distinction between the role and the occupant is relative, at least as it refers to mathematics: the thing that is an object, from one point of view, is a place or a role from another. The idea that a structure provides a system in which the system itself is exemplified is important for our considerations. The reason why is that, if what Shapiro says is correct, then the mathematician, when considering a consistent (or coherent) axiomatic system, will not only have the guarantee of the existence of a corresponding structure, but of a system that instantiates that structure.

Now FBP will be applied to mathematical structures. Only to these? Yes, although Shapiro does not explicitly go on record in this respect. In any event, what distinguishes mathematical structures? That they are independent (*freestanding*)[10], that is to say that anything can occupy a given role in a mathematical structure. The requirements are only that certain relations be maintained between this object and others, and that these relations be formal. For example, any eleven people do not

[6] Shapiro (1997), p. 134.
[7] However, he does not use Quine's ontological criterion exactly, but proposes a variant.
[8] Ibid. Pp. 73-74.
[9] Ibid. P. 10.
[10] Shapiro, S. (1997), p. 100.

constitute a football team, it might be that they are not willing to play, or that they are on different pitches, etc. Being on the same pitch is a non-formal relation. On the other hand, each one of them could be the neutral element of a group, if he maintains a specific type of relationship with other individuals or objects. Shapiro gives a characterization of what should be understood here as 'formal'[11], but it is not necessary for us to deal with the subject.

The distinction between two types of structuralism is fundamental to our considerations[12]. Given that a structure is similar to a universal, it can be viewed in either a Platonic or an Aristotelian way. In the former case, it would be independent of the objects that occupy the places in the various systems that instantiate it. This is the *ante rem* point of view. Alternatively, it can be considered as not being anything independently of the systems that instantiate it. This is the *in re* point of view. The structuralism that adopts the former point of view is called mythical (by Dummett) or *ante rem* (by Shapiro). The second type of structuralism adopts the latter perspective and is called eliminative or *in re* (by both Parsons and Shapiro). In this case, when determining the semantic value of a mathematical statement, it is not taken literally (*at face value*). Its apparent singular terms mask implicit bounded variables that range over the objects of systems that have this structure. Thus, if the semantics of an arithmetical statement ϕ is given by its paraphrase ϕ':

'for any system S, if S exemplifies the natural-number structure, the ϕ[S]

where ϕ[S] is obtained from ϕ by interpreting the nonlogical terminology and restricting the variables to the objects in S' This structuralism has two variants. The first, the ontological option, supposes that there is a background ontology that provides the objects of the various systems. This option does not interest us for the purposes of our study as it does not suppose that the structures exist and, it does suppose instead that there is a set of mathematical objects whose existence was not proven by consistency, but is already given in some other way.

The second type of *in re* structuralism is the modal option. In this option, the paraphrase of an arithmetical statement S is given as before, except that we attach the word 'possible' to 'system'. The arithmetician deals with possible systems that would instantiate the structure of the natural numbers. Here, FBP becomes 'the systems that satisfy a consistent axiomatic system exist'. But Hellman who developed this option, did not maintain FBP, nor was it plausible for him to do so. The defender of modal structuralism does not wish to annul modal distinctions. He precisely avoids the problems of the mathematical existence by appealing to the notion of possibility. In consequence, this version does not concern us here.

Shapiro favours the *ante rem* option of structuralism and it is for this option that he offers FBP as a criterion (or perhaps as a definition) of mathematical existence. In this case, the structures (characterized by consistent axiomatic systems) exist, whether they are instantiated in a system or not. Mathematical statements are interpreted in the 'places-are-objects' way (that is to say literally). Additionally, as we saw, each structure instantiates itself.

Notice that in the option adopted by Shapiro, mathematical objects are relative to a theory. When adding an independent axiom to a system we are implicitly changing the meaning of the defined terms. Now, does the same thing happen when proving a theorem? Apparently not, because for Shapiro logic is formal and therefore neutral to

[11] Shapiro, S. (1997), p. 99. This is the Tarskian definition.
[12] Ibid, pp. 84-90.

the subject. Let us illustrate this difference with an example. Let us suppose that there is a predicate A(x) in an axiomatic theory T. Let us considers two cases. In the first, a mathematician proves '(∃x)A(x)' with the axioms of T. In the second, a mathematician proves that it is not possible to prove '~(∃x)A(x)'and that therefore the theory T'= T∪{(∃x)A(x)} is consistent. Have they proven the same thing, that is, the existence of an object that satisfies 'A(x)'? No, because the meaning of A is given, in one case by the axioms of T and, in the other, by the axioms of T'.

By 'logic', Shapiro understands classical second order logic. What does he base this choice on? As regards 'second order', he bases it on the fact that mathematicians distinguish between standard models and other models, which would not be possible if his language was first order: 'Mathematicians themselves commonly make and exploit the distinction, and I presume that they are not deluding themselves. In the case of arithmetic, either informal resources go beyond those captured in formal logic, or we have a sufficient grasp of the second-order induction axiom. That is, we understand the second order quantifier well enough to see that all models of arithmetic are categorical.[13]'

Now, in second order logic, existence is not guaranteed by consistency . The example given by Shapiro is the following[14]: Let P be the conjunction of the axioms of (second order) arithmetic. Take S=P+¬G, where G is the sentence that states the consistency of P. S is consistent (by Gödel), but has no models. Clearly S is not a implicit coherent definition of a structure, in spite of its deductive consistency. Defining 'coherence' as the existence of a structure in the theory of structures does not work because we do not know if the latter is coherent, in addition to the fact that that one of the axioms invokes the notion of coherence. Shapiro thinks[15] that there is no way out of the circle and takes 'coherence'' as an intuitive, primitive notion. But he defends his choice, we are not completely in the dark with respect to this undefined notion: 'coherence' can at least be 'completely' explained. Shapiro says that the set-theoretical theory notion of satisfiability is a good mathematical model of 'coherence', because it captures much of the structure of coherence. According to him, we are in a situation analogous to Church's thesis.

Now we have sufficiently clarified each of the terms that appear in the version of FBP that Shapiro defends, that is, that mathematical objects are linked to structures and that a mathematical structure exists if a coherent axiomatization of it exists.

Let us now look at an argument in favour of FBP. Although Shapiro has said that the semantics of model theory is an suitable tool for realism, he also recognizes that, on its own, it does not say anything about the problem of what connects a name with its denotation. It only determines the relations between the conditions of truth, the extensions of predicates and the extensions of logical terminology. 'model theory is thus a functional (or structural) definition of these semantic terms… As far as the model-theoretic scheme goes, it does not matter how this "reference" is to be accomplished or whether it can be accomplished in accordance with some theory or other.' In model theory, the notion of 'reference' or 'designation' is primitive. We need to supplement it with an appropriate theory of reference to have 'a decent approximation of the truth conditions of those natural-language sentences that come closest to the formulas in a formal language.'

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[13] Shapiro (1977) p. 133.
[14] Ibid. P. 135
[15] Ibid. P. 135

And the decisive step comes here: 'Because mathematics is the science of structure, the "schematic" or structural semantic notions of model theory are all that we need. The details of the correct account of reference to physical objects are irrelevant... As noted, the student already understand reference and quantification, at least schematically. She may not know that there are any models of the theory, but she does grasp what it would be for a system to be such a model. Now, because the theory is categorical and coherent, all its models share a common structure. The suggestion of this book is that we think of real analysis as being about that very structure. Its variables range over the places of that structure, and its singular terms refer to some of those places. Knowing what it would be for a system to be a model of the axioms is to know what the real analysis structure is. Schematic knowledge about how language works leads to knowledge about structures. We end up with a model-theoretic interpretation of analysis. The variables range over the places of a structure... The rest is familiar model theoretic semantics. Once we realize what the ontology is, we have realism in ontology... If we insist on categorical characterizations of nonalgebraic theories, then we also have realism in truth value.'[16]

I have quoted this passage extensively because I believe that this is where we find our author's main argument in favour of his version of FBP. Shapiro had already said that the main means of access to the knowledge of mathematical structures is given by language. Now he adds an ontological conclusion. The idea, if I understand this correctly, is that model theory gives us no more than an outline, a characterization of the structure of semantic concepts. For example, independently of how we explain reference or how this is achieved, 'F(a)' is true if that to which 'a' refers has the property to which 'F' refers or which 'F' expresses. Here we only see how 'reference' and 'truth' are related to one another, for which reason we are still required to propose a theory that explains at least one of these semantic concepts. In an ordinary theory such an explanation is necessary if we wish to have the guarantee that the terms really refer, that is to say, that they designate things that exist. Of course, we could also have a semantic theory that explains how reference takes place, but does not assure us that, in a specific case, we have the knowledge that the singular terms truly refer. What Shapiro says is that the case of mathematics is different. There we can do without a theory of reference. The student who has a (coherent and categorical) axiomatic theory for Analysis will not initially have any guarantee that a system that satisfies those axioms exists, but he knows what that would mean. The important thing is that he does not require anything else. Understanding what it would be for a system to satisfy that structure places the structure itself before him and, since mathematics is not interested in systems but in structure, the mathematician is guaranteed having the object of his study complete. Additionally, according to Shapiro's version, as each structure U provides a system that satisfies U, he also has the guarantee that such a system exists. Supposing that that is our author's argument, how does coherence come in here? Perhaps it could be sustained that if the axiomatic theory is incoherent, the student does not know what it would be for a system to satisfy the corresponding structure because, in fact, there is no structure being described, even though he believes that he does indeed know it. And vice versa, if the axiomatic theory is coherent the student can understand what it would mean for a system to satisfy it, although he believes that it is not coherent.

We can see why FBP is only valid for mathematical structures. In other disciplines, we are interested in the systems and the essence of objects beyond their

[16] Ibid. P. 140

structural properties. If a structure is not mathematical, (non-formal) content relationships that are not captured in mathematical axioms are very much part of it.

The argument would not convince a nominalist, nor anyone in favour of other forms of structuralism. The case is similar to the dispute between nominalism and universalism. Let us consider a defender of this school who sketched out the following argument; now we are only interested in concepts and let us suppose that we have a consistent definition of a concept. As a result, we do not know if some object falls within the concept, but we do know what it would be for the concept to include an object. Therefore, we do not need anything else and we can conclude that the concept exists. The analogy is clear, a structure is a conceptual net: each gap in the structure is a kind of concept (although not independent from the other concepts in the structure) that is applied to objects of diverse systems. Anyone in favour of the *ante rem* option considers those concepts (defined only by their mutual interrelations) as the objects of his theory and, if the theory is coherent, we understand what it would be for a system to be an example of that structure. That does not mean that such a system exists nor, even if it did, that we have proved the independent existence of the structure. In fact, Hellman has criticized[17] Shapiro's step from 'they share a common structure' (possible systems, in the version of the modal structuralist) to 'there is a structure shared by all the systems…'

It is clear from the paragraph quoted that Shapiro does not identify coherence with the existence of a system satisfying the corresponding structure (although he has said that satisfiability is a good **explicatum** of the notion of coherence). Had he done so, it would still not be clear how that would have proved a nominalist wrong.

Now, it could be that Shapiro is not understanding 'existence' in a strong sense. Burgess argues[18] that what distinguishes the ingenuous realist from the philosophical realist is the step, for example, of:

There is a prime number greater than a googolplex

to

There is a number that satisfies 'x is prime and bigger than a googolplex'

Tarskian semantics makes that step trivial which is why Burgess believes Shapiro defends the notion that model-theoretic or Tarskian semantics is the essence of realism. Nevertheless, Burgess wonders, and he is unable to find the answer in Shapiro's text, if the truth should be understood here in the **decitational** sense or in a robust sense; and if 'existence' should be understood in Carnap's internal sense or in a stronger sense. Nevertheless, if it were the former, there would not be much distinction between ingenuous realism and philosophical realism. In any event, the cited version of FBP would not be very interesting.

Shapiro says that the argument in favour of realism is an inference towards the best explanation. This would seem to allude to another proof of FBP. What is this? Chihara says: 'I cannot find in Shapiro's book anything like an explicit argument for believing in the kind of abstract forms or structures that he postulates. However, he does make an implicit case for accepting his *ante rem* view of mathematics. His basic strategy is to undermine the main nominalistic rivals to his realistic account of mathematics and then to argue that his is the most perspicuous account of mathematics that is available.'[19] Apparently, to determine if this argument serves his ends, it would

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[17] Hellman, G. (1999), p. 925.
[18] Burgess, J. (1999)
[19] Chihara (2004), p. 71

be necessary to examine, as Chihara does, if *ante rem* structuralism offers the best explanation of mathematics and, in particular, if criticisms of other positions are well-founded. However, I believe that the question is simpler. It can be argued that the argument is circular. Indeed, Shapiro maintains that the three versions of structuralism are equivalent because they face problems of the same degree of difficulty. Let us concentrate for the time being on modal structuralism. This school also faces the problem of how to define 'consistency' or 'logical possibility'. He will have to take it as undefined (as his *ante rem* colleague did) and he will have to explain how we know that an axiomatic system is consistent without appealing to knowledge of abstract mathematical objects. In spite of this equivalence, Shapiro opts for the *ante rem* option 'as the most perspicuous'. Let us suppose that somebody has nominalistic tendencies (whether because he likes 'desert landscapes', like Quine, or because he is convinced by Benacerraf's argument, like Field) then he will think that FBP is false and that eliminative or modal structuralism is preferable to the *ante rem* version in spite of the equivalent difficulties that they face, precisely because it does not imply FBP.

The conclusion is that Shapiro does not have a solid argument in favour of FBP that will convince his rivals.

Conclusion

How might one argue in favour of FBP? Let us compare the case with that of the principle of indispensability. This also allows the step from the recognition of a methodological virtue to an existential statement, since it affirms the existence of the mathematical objects assumed indispensably by our best theory of the world. An important strength of Quine's argument is that he gives the same explanation for our attribution of existence to ordinary objects: 'By bringing together scattered sense events and treating them as perceptions on one object, we reduce the complexity of our stream of experience to a manageable conceptual simplicity... we associate an earlier and later round sensum with the same so-called penny, or with two different so-called pennies, in obedience to the demands of maximum simplicity in our total world-picture.'[20] We say that ordinary things or atoms or mathematical objects exist because postulating them simplifies the flow of our sense experience, that is to say, because they are part of our best theory of the world. They are myths or postulates, but we have no reason to deny the reality of atoms or sets if we accept the reality of the chair we are sitting on. An equivalent way of expressing this thesis is to say that only the objects postulated by our best theory of the world exist. And this is a central element in Quinean philosophy.

Shapiro objects that Quine's argument leaves the problem of applicability unsolved, but that is not so for, as Balaguer says, the mystery resulted from 'an inexplicable *correlation* between the mathematical domain and the physical world'. Nevertheless, in Quine's explanation, that mathematical domain did not exist before being applied. Given that, when designing it, it was found satisfactory in its application, we postulate it as being real. The most effective theory was declared true and its objects real. Now, for FBP we do not have a similar framework.

Shapiro has not provided a philosophical framework in which to justify the most extreme form of Platonism, or, at minimum, make it plausible. FBP does not provide a solution to Benacerraf's challenge.

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[20] (Quine, 1948) p. 17.