3. Rewrite these using dots:

$$(p(q \vee r) \supset s) \equiv (pq \supset s)(pr \supset s),$$

$$-(p \vee q)(r \vee s) \supset -(p \vee q)s.$$

4. Rewrite this using parentheses:

$$p \supset . q \vee r . p \vee qs :\equiv:. \bar{q}\bar{r} \vee . \bar{p} . \bar{q} \vee \bar{s} :\supset \bar{p}.$$

§5. TRUTH-VALUE ANALYSIS

In §2 a compound was said to be a truth function of its components when its truth value is determined by those of the components; and it was then observed that conjunction and negation constitute an adequate notation for truth functions. In view of this latter circumstance it is natural and convenient hereafter to conceive the notion of truth function in a purely notational way: the *truth functions of* given components are all the compounds constructed from them by means exclusively of conjunction and negation (and the dispensable further connectives '$\vee$', '$\supset$', '$\equiv$'). Thus '$\bar{p}$' is a truth function of 'p', and '$-(p \vee \bar{r}. \equiv pq) \supset r$' is a truth function of 'p', 'q', and 'r'. We also count 'p' itself a truth function of 'p'.

A truth function of letters 'p', 'q', etc., is strictly speaking not a statement, of course, since the letters are themselves not actual statements but mere dummies in place of which any desired statements may be imagined. Hereafter the letters 'p', 'q', etc., and all truth functions of them will be called *schemata* (singular: schema). More specifically they will be called *truth-functional schemata* when it becomes necessary to distinguish them from schemata involving logical devices of other than truth-functional kind. Schemata are logical diagrams of statements; the letters 'p', 'q', etc., by supplanting the component clauses of a statement, serve to blot out all the internal matter which is not germane to the broad outward structures with which our logical study is concerned.

By *interpretation* of the letter 'p' (or 'q', etc.) may be meant specification of an actual statement which is to be imagined in place of the letter. By interpretation of 'p' may also be meant simply specification of a truth value for 'p'. The two senses of 'interpreta-

tion' can be used pretty interchangeably because each actual statement S has a specific truth value (known or unknown) and that truth value is all that matters to the truth value of any truth function of S.

A convenient graphic method of imposing interpretations, of the second of the above varieties, is simply to supplant the letters in a schema by the mark '$\top$' for truths and '$\bot$' for falsehoods.[1] Computing then directly with these marks, we can quickly determine what truth value the whole schema takes on under the imposed interpretations. Thus, suppose our problem is to determine the truth value of the schema '$-(pq \vee \bar{p}\bar{q})$' for the case where '$p$' is interpreted as true and 'q' as false. We simply put '$\top$' for 'p' and '$\bot$' for 'q' in the schema, getting '$-(\top\bot \vee \bar{\top}\bar{\bot})$'. But, since '$\bar{\top}$' reduces to '$\bot$' and '$\bar{\bot}$' to '$\top$', this becomes '$-(\top\bot \vee \bot\top)$'. Further, since a conjunction with false component is false, '$\top\bot$' reduces to '$\bot$' and so does '$\bot\top$'. So the whole is now down to '$-(\bot \vee \bot)$'. But, an alternation of falsehoods being false, '$\bot \vee \bot$' reduces to '$\bot$'; the whole thus becomes '$\bar{\bot}$', or '$\top$'. This outcome means that our original schema '$-(pq \vee \bar{p}\bar{q})$' comes out true when '$p$' is interpreted as true and 'q' as false.

The process whereby '$-(\top\bot \vee \bar{\top}\bar{\bot})$' was reduced to '$\top$' will be called *resolution*. The simplest of the steps involved in resolution, viz. reduction of '$\bar{\top}$' to '$\bot$' and of '$\bar{\bot}$' to '$\top$', will always be tacit hereafter; we shall never write '$\bar{\top}$' nor '$\bar{\bot}$', but immediately '$\bot$' and '$\top$', as if the notation of negation as applied to '$\top$' and '$\bot$' consisted simply in inverting. The other steps of resolution illustrated in the above example were reduction of '$\top\bot$', '$\bot\top$', and '$\bot \vee \bot$' to '$\bot$'. These steps, and all further ones for which there might be occasion in other examples, may conveniently be codified in the form of eight *rules of resolution:*

(i) *Delete '$\top$' as component of conjunction.* (Thus '$\top\top\top$' reduces to '$\top\top$' and thence to '$\top$'; '$\bot\top$' reduces to '$\bot$'; etc. Reason: a conjunction with a true component is true or false according as the rest of it is true or false.)

(ii) *Delete '$\bot$' as component of alternation.* (Thus '$\bot \vee \bot \vee \bot$' re-

[1] We need not fumble for a pronunciation of '$\bot$' coördinate with the pronunciation 'tee' of '$\top$', for the words 'true' and 'false' themselves are short enough to serve conveniently as pronunciations of the two signs. Before deploring my preference of '$\bot$' to the initial 'F' of 'false', note the urgent need of 'F' for other purposes in Parts II–IV.

duces to '$\bot \vee \bot$' and thence to '$\bot$'; '$\bot \vee \top$' reduces to '$\top$'; etc. Reason: an alternation with a false component is true or false according as the rest of it is true or false.)

(iii) *Reduce a conjunction with* '$\bot$' *as component to* '$\bot$'.

(iv) *Reduce an alternation with* '$\top$' *as component to* '$\top$'.

(v) *If a conditional has* '$\top$' *as antecedent or consequent, drop the antecedent.* (Thus '$\top \supset \top$' and '$\bot \supset \top$' reduce to '$\top$', and '$\top \supset \bot$' reduces to '$\bot$'. Reason: a conditional with true antecedent is true or false according as the consequent is true or false; and a conditional with true consequent is true.)

(vi) *If a conditional has* '$\bot$' *as antecedent or consequent, negate the antecedent and drop the consequent.* (Thus '$\bot \supset \top$' and '$\bot \supset \bot$' reduce to '$\top$', and '$\top \supset \bot$' reduces to '$\bot$'. Reason: a conditional with false antecedent is true, and a conditional with false consequent is true or false according as the antecedent is false or true.)

(vii) *Drop* '$\top$' *as component of a biconditional.* (Thus '$\top \equiv \top$' reduces to '$\top$', and '$\top \equiv \bot$' and '$\bot \equiv \top$' reduce to '$\bot$'.)

(viii) *Drop* '$\bot$' *as component of a biconditional and negate the other side.* (Thus '$\bot \equiv \bot$' reduces to '$\top$', and '$\top \equiv \bot$' and '$\bot \equiv \top$' reduce to '$\bot$'.)

Set up according to these rules, our original example of resolution amounts to no more than this:

$-(\top\bot \vee \bot\top)$	
$-(\bot \vee \bot)$	(changing '$\top\bot$' and '$\bot\top$' each to '$\bot$' by (i) or (iii))
$\top$	(changing '$\bot \vee \bot$' to '$\bot$' by (ii))

Turning to a more elaborate example, let us determine the truth value of '$pq \vee \bar p \bar r \,.\supset.\, q \equiv r$' for the case where '$p$' and '$q$' are interpreted as false and 'r' as true.

$\bot\bot \vee \top\bot \,.\supset.\, \bot \equiv \top$	
$\bot\bot \vee \top\bot \,.\supset\, \bot$	(changing '$\bot \equiv \top$' to '$\bot$' by (vii) or (viii))
$-(\bot\bot \vee \top\bot)$	(by (vi))
$-(\bot \vee \bot)$	(by (iii) twice)
$\top$	(changing '$\bot \vee \bot$' to '$\bot$' by (ii))

Thus '$pq \vee \bar p \bar r \,.\supset.\, q \equiv r$' comes out true when false statements are put for 'p' and 'q' and a true one for 'r'.

Let us feign contact with reality by considering an actual statement of the form '$pq \vee \bar p \bar r \,.\supset.\, q \equiv r$':

(1) If either the resident and the deputy resident both resign or the resident neither resigns nor exposes the *chargé d'affaires*, in either case the deputy resident will resign if and only if the resident exposes the *chargé d'affaires.*

What we have found is that (1) comes out true in the case where neither the resident nor the deputy resident resigns and the resident exposes the *chargé d'affaires.*

We have evaluated the schema '$pq \vee \bar p \bar r \,.\supset.\, q \equiv r$' for one interpretation: 'p' and 'q' as false and 'r' as true. There remain seven other interpretations that might be considered: 'p', 'q', and 'r' all true, 'p' and 'q' true and 'r' false, 'p' and 'r' true and 'q' false, and so on. The eight cases can be systematically explored, with evaluation of the schema for each case, by the following method. First we put '$\top$' for 'p', leaving 'q' and 'r' unchanged, and make all possible resolutions by (i)–(viii):

$\top q \vee \bot \bar r \,.\supset.\, q \equiv r$	
$q \vee \bot \bar r \,.\supset.\, q \equiv r$	(changing '$\top q$' to 'q' by (i))
$q \vee \bot \,.\supset.\, q \equiv r$	(changing '$\bot \bar r$' to '$\bot$' by (iii))
$q \supset.\, q \equiv r$	(changing '$q \vee \bot$' to 'q' by (ii))

Then we put '$\top$' for 'q' in this result and resolve further:

$\top \supset.\, \top \equiv r$	
$\top \equiv r$	(by (v))
r	(by (vii))

We have now found that whenever 'p' and 'q' are both interpreted as true, our original schema resolves to 'r'—hence becomes true or false according as 'r' is true or false. This disposes of two of the eight cases. Next we return to our intermediate result '$q \supset.\, q \equiv r$' and put '$\bot$' for 'q':

$\bot \supset.\, \bot \equiv r$	
$\top$	(by (vi))

This shows that our original schema comes out true whenever 'p' is interpreted as true and 'q' as false, regardless of 'r'. This disposes of two more of our eight cases. Now we go all the way back to our original schema and put '$\bot$' for 'p':

$$\bot q \vee \top\bar{r} \;.\!\supset\!.\; q \equiv r$$
$$\bot \vee \top\bar{r} \;.\!\supset\!.\; q \equiv r \qquad \text{(by (iii))}$$
$$\top\bar{r} \supset . \, q \equiv r \qquad \text{(by (ii))}$$
$$\bar{r} \supset . \, q \equiv r \qquad \text{(by (i))}$$

Putting '$\top$' for 'r' here and resolving further, we have:

$$\bot \supset . \, q \equiv \top$$
$$\top \qquad \text{(by (vi))}$$

This shows that our original schema comes out true whenever 'p' is interpreted as false and 'r' as true, regardless of 'q'. Two more cases are disposed of. Finally we go back to '$\bar{r} \supset . \, q \equiv r$' and put '$\bot$' for '$r$':

$$\top \supset . \, q \equiv \bot$$
$$q \equiv \bot \qquad \text{(by (v))}$$
$$\bar{q} \qquad \text{(by (viii))}$$

So whenever 'p' and 'r' are both interpreted as false, our schema resolves to '$\bar{q}$'—hence becomes false or true according as 'q' is interpreted as true or false.

The foregoing analysis might conveniently have been carried out in a single array as follows:

$$pq \vee \bar{p}\bar{r} \;.\!\supset\!.\; q \equiv r$$
$$\top q \vee \bot\bar{r} \;.\!\supset\!.\; q \equiv r \qquad\qquad \bot q \vee \top\bar{r} \;.\!\supset\!.\; q \equiv r$$
$$q \vee \bot\bar{r} \;.\!\supset\!.\; q \equiv r \qquad\qquad \bot \vee \top\bar{r} \;.\!\supset\!.\; q \equiv r$$
$$q \vee \bot \;.\!\supset\!.\; q \equiv r \qquad\qquad \top\bar{r} \supset . \, q \equiv r$$
$$q \supset . \, q \equiv r \qquad\qquad \bar{r} \supset . \, q \equiv r$$
$$\top \supset . \top \equiv r \qquad \bot \supset . \bot \equiv r \qquad \bot \supset . \, q \equiv \top \qquad \top \supset . \, q \equiv \bot$$
$$\top \equiv r \qquad\qquad \top \qquad\qquad \top \qquad\qquad q \equiv \bot$$
$$r \qquad\qquad\qquad\qquad\qquad\qquad \bar{q}$$
$$\top \quad \bot \qquad\qquad\qquad\qquad\qquad\qquad \bot \quad \top$$

This is called a ***truth-value analysis***. The general method may be summed up as follows. We make a grand dichotomy of cases by putting first '$\top$' and then '$\bot$' for some chosen letter, say 'p'. The expressions thus formed are the respective headings of a bipartite analysis.

Then we resolve both expressions, by (i)–(viii), until we end up with '$\top$' or '$\bot$' or some schema. If a schema results, we then proceed to develop, under that schema, a new bipartite analysis with respect to a chosen one of its letters. We continue thus until all end results are single marks — '$\top$' or '$\bot$'. Each end result shows what truth value the original schema will take on when its letters are interpreted according to the marks which have there supplanted them.

Actually all intermediate steps of resolution are so obvious, and so readily reconstructed at will, that they may hereafter be left to the imagination. Thus the above truth-value analysis would in future be condensed as follows:

$$pq \vee \bar{p}\bar{r} \;.\!\supset\!.\; q \equiv r$$
$$\top q \vee \bot\bar{r} \;.\!\supset\!.\; q \equiv r \qquad\qquad \bot q \vee \top\bar{r} \;.\!\supset\!.\; q \equiv r$$
$$q \supset . \, q \equiv r \qquad\qquad \bar{r} \supset . \, q \equiv r$$
$$\top \supset . \top \equiv r \qquad \bot \supset . \bot \equiv r \qquad \bot \supset . \, q \equiv \top \qquad \top \supset . \, q \equiv \bot$$
$$r \qquad\qquad \top \qquad\qquad \top \qquad\qquad \bar{q}$$
$$\top \quad \bot \qquad\qquad\qquad\qquad\qquad\qquad \bot \quad \top$$

There is no need always to choose 'p' as the first letter for which to put '$\top$' and '$\bot$'. It is better to choose the letter which has the most repetitions, if repetitions there be, and to adhere to this plan also at each later stage. Thus it was, indeed, that whereas in the second stage on the left side of the above analysis 'q' was chosen for replacement by '$\top$' and '$\bot$', on the other hand in the second stage on the right side 'r' was chosen. This strategy tends to hasten the disappearance of letters, and thus to minimize work.

A method of truth-value analysis which has been usual in the literature since 1920–21 (Łukasiewicz, Post, Wittgenstein) is that of *truth tables*. Under this method all combinations of truth values for the letters of a schema are listed, and for each combination the truth value of the schema is computed by a process of reasoning tantamount to what I have called resolution. This method has the shortcoming of cumbersomeness when many letters are involved. Where 'p', 'q', and 'r' are concerned, eight combinations of truth values have to be dealt with. Where four letters are concerned, as in the example three pages hence, sixteen combinations have to be dealt with. The advantage of the technique presented in the present pages is that the various combinations of truth values concerned tend to group themselves to form a smaller number of cases.