APPENDIX II

THE STRUCTURE OF THE SYSTEM OF STRICT IMPLICATION[1]

The System of Strict Implication, as presented in Chapter V of *A Survey of Symbolic Logic* (University of California Press, 1918), contained an error with respect to one postulate. This was pointed out by Dr. E. L. Post, and was corrected by me in the *Journal of Philosophy, Psychology, and Scientific Method* (XVII [1920], 300). The amended postulates (set A below) compare with those of Chapter VI of this book (set B below) as follows:

[1] This appendix is written by Mr. Lewis, but the points demonstrated are, most of them, due to other persons.

Groups II and III, below, were transmitted to Mr. Lewis by Dr. M. Wajsberg, of the University of Warsaw, in 1927. Dr. Wajsberg's letter also contained the first proof ever given that the System of Strict Implication is not reducible to Material Implication, as well as the outline of a system which is equivalent to that deducible from the postulates of Strict Implication with the addition of the postulate later suggested in Becker's paper and cited below as C11. It is to be hoped that this and other important work of Dr. Wajsberg will be published shortly.

Groups I, IV, and V are due to Dr. William T. Parry, who also discovered independently Groups II and III. Groups I, II, and III are contained in his doctoral dissertation, on file in the Harvard University Library. Most of the proofs in this appendix have been given or suggested by Dr. Parry.

It follows from Dr. Wajsberg's work that there is an unlimited number of groups, or systems, of different cardinality, which satisfy the postulates of Strict Implication. Mr. Paul Henle, of Harvard University, later discovered another proof of this same fact. Mr. Henle's proof, which can be more easily indicated in brief space, proceeds by demonstrating that any group which satisfies the Boole-Schröder Algebra will also satisfy the postulates of Strict Implication if $\Diamond p$ be determined as follows:

$$\Diamond p = 1 \text{ when and only when } p \neq 0;$$

$$\Diamond p = 0 \text{ when and only when } p = 0.$$

This establishes the fact that there are as many distinct groups satisfying the postulates as there are powers of 2, since it has been shown by Huntington that there is a group satisfying the postulates of the Boole-Schröder Algebra for every power of 2 ("Sets of Independent Postulates for the Algebra of Logic," *Trans. Amer. Math. Soc.*, V [1904], 309).

The proof of (14), p. 498, is due to Y. T. Shen (Shen Yuting).

A1. $p\,q \,.\prec.\, q\,p$
A2. $q\,p \,.\prec.\, p$
A3. $p \,.\prec.\, p\,p$
A4. $p(q\,r) \,.\prec.\, q(p\,r)$
A5. $p \prec \sim(\sim p)$
A6. $p \prec q \,.\, q \prec r :\prec.\, p \prec r$
A7. $\sim\Diamond p \prec \sim p$
A8. $p \prec q \,.\prec.\, \sim\Diamond q \prec \sim\Diamond p$

B1. $p\,q \,.\prec.\, q\,p$
B2. $p\,q \,.\prec.\, p$
B3. $p \,.\prec.\, p\,p$
B4. $(p\,q)r \,.\prec.\, p(q\,r)$
B5. $p \prec \sim(\sim p)$
B6. $p \prec q \,.\, q \prec r :\prec.\, p \prec r$
B7. $p \,.\, p \prec q :\prec.\, q$
B8. $\Diamond(p\,q) \prec \Diamond p$
B9. $(\exists\, p, q) : \sim(p \prec q) \,.\, \sim(p \prec \sim q)$

The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many other points concerning the structure of Strict Implication, will be facilitated by consideration of the following groups. Each of these is based upon the same matrix for the relation $p\,q$ and the function of negation $\sim p$. (This is a four-valued matrix which satisfies the postulates of the Boole-Schröder Algebra.) The groups differ by their different specification of the function $\Diamond p$. We give the fundamental matrix for $p\,q$ and $\sim p$ in the first case only. The matrix for $p \prec q$, resulting from this and the particular determination of $\Diamond p$, is given for each group:

Group I

	$p\,q$	q: 1	2	3	4	$\sim p$
p	1	1	2	3	4	4
	2	2	2	4	4	3
	3	3	4	3	4	2
	4	4	4	4	4	1

$\Diamond$	$\prec$	1	2	3	4
1	1	2	4	4	4
1	2	2	2	4	4
1	3	2	4	2	4
3	4	2	2	2	2

Group II

$\Diamond$	$\prec$	1	2	3	4
1	1	1	4	3	4
2	2	1	1	3	3
1	3	1	4	1	4
4	4	1	1	1	1

Group III

$\Diamond$	$\prec$	1	2	3	4
1	1	1	4	4	4
1	2	1	1	4	4
1	3	1	4	1	4
4	4	1	1	1	1

Group IV

$\Diamond$	$\prec$	1	2	3	4
2	1	1	3	3	3
2	2	1	1	3	3
2	3	1	3	1	3
4	4	1	1	1	1

Group V

$\Diamond$	$\prec$	1	2	3	4
1	1	2	4	3	4
2	2	2	2	3	3
1	3	2	4	2	4
3	4	2	2	2	2

The 'designated values,' for all five groups, are 1 and 2; that is, the group is to be taken as satisfying any principle whose values, for all combinations of the values of its variables, are confined to 1 and 2. (In Groups II, III, and IV, 1 alone might be taken as the designated value: but in that case it must be remembered that, since

$$(\exists\, p, q) : \sim(p \prec q) \,.\, \sim(p \prec \sim q) :.\, = \,:.\, \sim[(p, q) : p \prec q \,.\vee.\, p \prec \sim q],$$

B9 would be satisfied *unless* $p \prec q \,.\vee.\, p \prec \sim q$ *always* has the value 1. It is simpler to take 1 and 2 both as designated values; in which case B9 is satisfied if and only if $\sim(p \prec q) \,.\, \sim(p \prec \sim q)$ has the value 1 or the value 2 for some combination of the values of p and q.)

All of these groups satisfy the operations of 'Adjunction,' 'Inference,' and the substitution of equivalents. If P and Q are functions having a designated value, then $P\,Q$ will have a designated value. If P has a designated value, and $P \prec Q$ has a designated value, then Q will have a designated value. And if $P = Q$—that is, if $P \prec Q \,.\, Q \prec P$ has a designated value—then P and Q will have the same value, and for any function f in the system, $f(P)$ and $f(Q)$ will have the same value.

The following facts may be established by reference to these groups:

(1) The system, as deduced from either set of postulates, is consistent. Group I, Group II, and Group III each satisfy all postulates of either set. For any one of these three groups, B9 is satisfied by the fact that $\sim(p \prec q) \,.\, \sim(p \prec \sim q)$ has a designated value when $p = 1$ and $q = 2$, and when $p = 1$ and $q = 3$.

(2) The system, as deduced from either set, is not reducible to Material Implication. For any one of the five groups,

$\sim(p \sim q) . \prec . p \prec q$ has the value 3 or 4 when $p = 1$ and $q = 2$. None of the 'paradoxes' of Material Implication, such as $p . \supset . q \supset p$ and $\sim p . \supset . p \supset q$, will hold for any of these groups if the sign of material implication, $\supset$, is replaced by $\prec$ throughout.

(3) The Consistency Postulate, B8, is independent of the set (B1–7 and B9) and of the set A1–7. Group V satisfies B1–7, and satisfies $\sim(p \prec q) . \sim(p \prec \sim q)$ for the values $p = 1$, $q = 2$. It also satisfies A1–7. But Group V fails to satisfy B8: B8 has the value 4 when $p = 2$ and $q = 3$, and when $p = 2$ and $q = 4$.

(4) Similarly, A8 is independent of the set A1–7, and of the set (B1–7 and B9). For Group V, A8 has the value 4 when $p = 1$ and $q = 3$, and when $p = 2$ and $q = 3$.

(5) Postulate B7 is independent of the set (B1–6 and B8, 9), and of the set (A1–6 and A8). Group IV satisfies B1–6, B8, and B9, and satisfies A1–6 and A8. But for this group, B7 has the value 3 when $p = 1$ and $q = 2$, and for various other combinations of the values of p and q.

(6) Similarly, A7 is independent of the set (A1–6 and A8) and of the set (B1–6 and B8, 9). For Group IV, A7 has the value 3 when $p = 1$ and when $p = 3$.

That the Existence Postulate, B9, is independent of the set B1–8, and of the set A1–8, is proved by the following two-element group, which satisfies B1–8 and A1–8:

$p\ q$	1	0	$\sim p$
1	1	0	0
0	0	0	1

	$\Diamond$	$\prec$	1	0
1	1		1	0
0	0		1	1

(This is, of course, the usual matrix for Material Implication, with the function $\Diamond p$ specified as equivalent to p.) For this group, $\sim(p \prec q) . \sim(p \prec \sim q)$ has the value 0 for all combinations of the values of p and q.

Dr. Parry has been able to deduce B2 from the set (B1 and B3–9). However, the omission of B2 from the postulate set of Chapter VI would have been incompatible with the order of exposition there adopted, since the Consistency Postulate is required for the derivation of B2. Whether with this exception the members of set B are mutually independent has not been fully determined.

The question naturally arises whether the two sets A1–8 and B1–8 are equivalent. I have discovered no proof but believe that they are not. B1–8 are all deducible from A1–8: and A1–7 are all deducible from B1–8. The question is whether A8 is deducible from B1–8. If it is not, then the system as deduced from the postulate set of Chapter VI, B1–9, is somewhat 'stricter' than as deduced in the *Survey* from set A.

The logically important issue here concerns certain consequences which enter the system when A8 is introduced. Both Dr. Wajsberg and Dr. Parry have proved that the principle

$$p \prec q . \prec : q \prec r . \prec . p \prec r$$

is deducible from A1–8. I doubt whether this proposition should be regarded as a valid principle of deduction: it would never lead to any inference $p \prec r$ which would be questionable when $p \prec q$ and $q \prec r$ are given premises; but it gives the inference $q \prec r . \prec . p \prec r$ whenever $p \prec q$ is a premise. Except as an elliptical statement for "$p \prec q . q \prec r : \prec . p \prec r$ and $p \prec q$ is true," this inference seems dubious.

Now as has been proved under (3) above, the Consistency Postulate, B8, is not deducible from the set (B1–7 and B9). Likewise the principle mentioned in the preceding paragraph is independent of the set (B1–7 and B9): Group V satisfies this set, but for that group the principle in question has the value 4 when $p = 1$, $q = 3$, and $r = 1$, as well as for various other values of p, q, and r. But Group V also fails to satisfy B8, as was pointed out in (3) above. If it should hereafter be discovered that the dubious principle of the preceding paragraph is deducible from the set B1–9, then at least it is not contained in the system deducible from the set (B1–7 and B9); and I should then regard that system—to be referred to hereafter as S1—as the one which coincides in its properties with the strict principles of deductive inference. As the reader will have noted, Chapter VI was so developed that the theorems belonging to this system, S1, are readily distinguishable from those which require the Consistency Postulate, B8.

The system as deduced either from set A or from set B leaves undetermined certain properties of the modal functions, $\Diamond p$, $\sim\Diamond p$, $\Diamond\sim p$, and $\sim\Diamond\sim p$. In view of this fact, Professor Oskar Becker [2] has proposed the following for consideration as further postulates, any one or more of which might be added to either set:

C10. $\sim\Diamond\sim p \prec \sim\Diamond\sim\ \sim\Diamond\sim p$ $\qquad \sim\Diamond\sim\ \sim\Diamond\sim p = \sim\Diamond\sim p$

C11. $\Diamond p \prec \sim\Diamond\ \sim\Diamond p$ $\qquad \Diamond p = \sim\Diamond\ \sim\Diamond p$

C12. $p \prec \sim\Diamond\ \sim\Diamond p$

(Becker calls C12 the "Brouwersche Axiom.")

When A1–8, or B1–9, are assumed, the second form in which C10 is given can be derived from the first, since the converse implication, $\sim\Diamond\sim\ \sim\Diamond\sim p \prec \sim\Diamond\sim p$, is an immediate consequence of the general principle, $\sim\Diamond\sim p \prec p$ (18·42 in Chapter VI). The second form of C11 is similarly deducible from the first.

An alternative and notationally simpler form of C10 would be

C10·1 $\Diamond\,\Diamond p \prec \Diamond p$ $\qquad \Diamond\,\Diamond p = \Diamond p$

(As before, the second form of the principle can be derived from the first; since the converse implication, $\Diamond p \prec \Diamond\,\Diamond p$, is an instance of the general principle $p \prec \Diamond p$, which is 18·4 in Chapter VI, deducible from A1–8, or B1–9.)

Substituting $\sim p$ for p, in C10·1, we have

$$\Diamond\,\Diamond\sim p \prec \Diamond\sim p \qquad \text{(a)}$$

$$\text{(a)} . = . \sim\Diamond\sim p \prec \sim\Diamond\ \Diamond\sim p . = . \sim\Diamond\sim p \prec \sim\Diamond\sim\ \sim\Diamond\sim p.$$

And substituting $\sim p$ for p in C10, we have

$$\sim\Diamond\sim(\sim p) \prec \sim\Diamond\sim\ \sim\Diamond\sim(\sim p) \qquad \text{(b)}$$

$$\text{(b)} . = . \sim\Diamond p \prec \sim\Diamond\sim\ \sim\Diamond p . = . \sim\Diamond p \prec \sim\Diamond\ \Diamond p . = . \Diamond\ \Diamond p \prec \Diamond p.$$

(The principles used in these proofs are 12·3 and 12·44 in Chapter VI.)

For reasons which will appear, we add, to this list of further postulates to be considered, the following:

C13. $\Diamond\,\Diamond p$

That is, "For every proposition p, the statement 'p is self-consistent' is a self-consistent statement."

[2] See his paper "Zur Logik der Modalitäten," *Jahrbuch für Philosophie und Phänomenologische Forschung*, XI (1930), 497–548.

Concerning these proposed additional postulates, the following facts may be established by reference to Groups I, II, and III, above, all of which satisfy the set A1–8 and the set B1–9:

(7) C10, C11, and C12 are all consistent with A1–8 and with B1–9 and with each other. Group III satisfies C10, C11, and C12.

(8) C10, C11, and C12 are each independent of the set A1–8 and of the set B1–9. For Group I, C10, C11, and C12 all fail to hold when $p = 3$.

(9) Neither C11 nor C12 is deducible from the set (A1–8 and C10) or from the set (B1–9 and C10). Group II satisfies C10; but C11 fails, for this group, when $p = 2$ or $p = 4$; and C12 fails when $p = 2$.

(10) C13 is consistent with the set A1–8 and with the set B1–9. Group I satisfies C13.

(11) C13 is independent of the set A1–8 and of the set B1–9, and of (A1–8 and C10, C11, and C12) or (B1–9 and C10, C11, and C12). Group III satisfies all these sets; but for this group, C13 fails when $p = 4$.

When A1–8, or B1–9, are assumed, the relations of C10, C11, and C12 to each other are as follows:

(12) C10 is deducible from C11. By C11 and the principle $\sim(\sim p) = p$,

$$\begin{aligned} \sim\Diamond\sim p &= \sim[\Diamond(\sim p)] = \sim[\sim\Diamond \sim\Diamond(\sim p)] = \Diamond[\sim\Diamond(\sim p)] \\ &= \sim\Diamond \sim\Diamond(\sim\Diamond \sim p) = \sim\Diamond[\sim\Diamond \sim\Diamond(\sim p)] \\ &= \sim\Diamond[\Diamond(\sim p)] = \sim\Diamond\{\sim[\sim(\Diamond\sim p)]\} = \sim\Diamond\sim \sim\Diamond\sim p. \end{aligned}$$

(13) C12 also is deducible from C11. By 18·4 in Chapter VI, $p \prec \Diamond p$; and this, together with C11, implies C12, by A6 or by B6.

(14) From C10 and C12 together, C11 is deducible. Substituting $\Diamond p$ for p in C12, we have

$$\Diamond p \prec \sim\Diamond \sim\Diamond \Diamond p. \qquad \text{(a)}$$

And by C10·1, $\sim\Diamond \sim\Diamond \Diamond p = \sim\Diamond \sim\Diamond p$. Hence (a) is equivalent to C11.

From (12), (13), and (14), it follows that as additional postulates to the set A1–8, or the set B1–9, C11 is exactly equivalent to C10 and C12 together. But as was proved in (9), the addition of C10 alone, gives a system in which neither C11 nor C12 is deducible.

Special interest attaches to C10. The set A1–8, or the set B1–9, *without* C10, gives the theorem

$$\sim\Diamond\sim\ \sim\Diamond\sim p\ .\ \prec :\ \sim\Diamond\sim p = p.$$

This is deducible from 19·84 in Chapter VI. It follows from this that if there should be some proposition p such that $\sim\Diamond\sim\ \sim\Diamond\sim p$ is true, then the equivalences

$$p = \sim\Diamond\sim p \qquad \text{and} \qquad \sim\Diamond\sim p = \sim\Diamond\sim\ \sim\Diamond\sim p$$

would hold for that particular proposition. And since, by 19·84 itself, all necessary propositions are equivalent, it follows that if there is *any* proposition p which is necessarily-necessary—such that $\sim\Diamond\sim\ \sim\Diamond\sim p$ is true—then *every* proposition which is necessary is also necessarily-necessary; and the principle stated by C10 holds universally. But as was proved in (8), this principle, $\sim\Diamond\sim p = \sim\Diamond\sim\ \sim\Diamond\sim p$, is not deducible from A1–8 or from B1–9. Hence the two possibilities, with respect to necessary propositions, which the system, as deduced from A1–8 or from B1–9, leaves open are: (a) that there exist propositions which are necessarily-necessary, and that for every proposition p, $\sim\Diamond\sim p = \sim\Diamond\sim\ \sim\Diamond\sim p$; and (b) that there exist propositions which are necessary—as 20·21 in Chapter VI requires—but *no* propositions which are *necessarily*-necessary. This last is exactly what is required by C13, $\Diamond\ \Diamond p$. Substituting here $\sim p$ for p, we have, as an immediate consequence of C13, $\Diamond\ \Diamond\sim p$. This is equivalent to the theorem "For every proposition p, 'p is necessarily-necessary' is false": $\Diamond\ \Diamond\sim p = \Diamond\sim\ \sim\Diamond\sim p = \sim(\sim\Diamond\sim\ \sim\Diamond\sim p)$ [by the principle $\sim(\sim p) = p$]. Thus C10 expresses alternative (a) above; and C13 expresses alternative (b). Hence as additional postulates, C10 and C13 are contrary assumptions.

(As deduced from A1–8, the system leaves open the further alternative that there should be no necessary propositions, or that the class of necessary propositions should merely coincide with the class of true propositions; but in that case the system becomes a redundant form of Material Implication.)

From the preceding discussion it becomes evident that there is a group of systems of the general type of Strict Implication and each distinguishable from Material Implication. We shall arrange these in the order of increasing comprehensiveness and decreasing 'strictness' of the implication relation:

S1, deduced from the set B1–7, contains all the theorems of Sections 1–4 in Chapter VI. It contains also all theorems of Section 5, in the form of T-principles, but not with omission of the T. This system does not contain A8 or the principle

$$p \prec q \,.\, \prec : q \prec r \,.\, \prec \,.\, p \prec r.$$

However, it does contain, in the form of a T-principle, any theorem which could be derived by using A8 as a principle of inference: because it contains

$$p \prec q \,.\, {\sim}\Diamond q : \prec \,.\, {\sim}\Diamond p;$$

and hence if (by substitutions) $p \prec q$ becomes an asserted principle, we shall have

$$T \,.\, {\sim}\Diamond q : \prec \,.\, {\sim}\Diamond p.$$

When the Existence Postulate, B9, is added, this system S1 contains those existence theorems which are indicated in Section 6 of Chapter VI as not requiring the Consistency Postulate, B8.

S2, deduced from the set B1–8, contains all the theorems of Sections 1–5 in Chapter VI, any T-principle being replaceable by the corresponding theorem without the T. When the Existence Postulate, B9, is added, it contains all the existence theorems of Section 6. Whether S2 contains A8 and the principle

$$p \prec q \,.\, \prec : q \prec r \,.\, \prec \,.\, p \prec r$$

remains undetermined. If that should be the case, then it will be equivalent to S3.

S3, deduced from the set A1–8, as in the *Survey*, contains all the theorems of S2 and contains such consequences of A8 as

$$p \prec q \,.\, \prec : q \prec r \,.\, \prec \,.\, p \prec r.$$

If B9 is added, the consequences include all existence theorems of S2.

For each of the preceding systems, S1, S2, and S3, any one of the additional postulates, C10, C11, C12, and C13, is consistent with but independent of the system (but C10 and C13 are mutually incompatible).

S4, deduced from the set (B1–7 and C10), contains all theorems of S3, and in addition the consequences of C10. A8 and B8 are deducible theorems. S4 is incompatible with C13. C11 and C12 are consistent with but independent of S4. If B9 be added, the consequences include all existence theorems of S2.

S5, deduced from the set (B1–7 and C11), or from the set (B1–7, C10, and C12), contains all theorems of S4 and in addition the consequences of C12. If B9 be added, all existence theorems of S2 are included. A8 and B8 are deducible theorems. S5 is incompatible with C13.

Dr. Wajsberg has developed a system mathematically equivalent to S5, and has discovered many important properties of it, notably that it is the limiting member of a certain family of systems. Mr. Henle has proved that S5 is mathematically equivalent to the Boole-Schröder Algebra (*not* the Two-valued Algebra), if that algebra be interpreted for propositions, and the function $\Diamond p$ be determined by:

$$\Diamond p = 1 \text{ when and only when } p \neq 0;$$
$$\Diamond p = 0 \text{ when and only when } p = 0.$$

In my opinion, the principal logical significance of the system S5 consists in the fact that it divides all propositions into two mutually exclusive classes: the intensional or modal, and the extensional or contingent. According to the principles of this system, all intensional or modal propositions are either necessarily true or necessarily false. As a consequence, for any modal proposition—call it p_m—

$$\Diamond(p_m) = (p_m) = \sim\Diamond\sim(p_m),$$
$$\text{and } \Diamond\sim(p_m) = \sim(p_m) = \sim\Diamond(p_m).$$

For extensional or contingent propositions, however, possibility, truth, and necessity remain distinct.

Prevailing good use in logical inference—the practice in mathematical deductions, for example—is not sufficiently precise and self-conscious to determine clearly which of these five systems expresses the acceptable principles of deduction. (The meaning of 'acceptable' here has been discussed in Chapter VIII.) The issues concern principally the nature of the relation of 'implies' which is to be relied upon for inference, and certain subtle questions about the meaning of logical 'necessity,' 'possibility' or 'self-consistency,' etc.—for example, whether C10 is true or false. (Professor Becker has discussed at length a number of such questions, in his paper above referred to.) Those interested in the merely mathematical properties of such systems of symbolic logic tend to prefer the more comprehensive and less 'strict' systems, such as S5 and Material Implication. The interests of logical study would probably be best served by an exactly opposite tendency.