QUANTUM GRAVITY SIGNALS FROM ALGEBRAIC STABILITY CONSIDERATIONS

C. Chryssomalakos and E. Okon

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apdo. Postal 70-543, 04510 México, D.F., MÉXICO, chryss@nuclecu.unam.mx, eliokon@nuclecu.unam.mx

ABSTRACT

We apply Lie algebra deformation theory to the problem of identifying the stable form of the quantum relativistic kinematical algebra. Three possible deformations are found, which introduce dimensionful constants. We also argue that, instead of positions, moments should serve as Lie algebra generators, leading to a radically different interpretation of the nature of the deformations.

Key words: Poincaré algebra deformations; Planck scale physics; invariant length scale.

1. INTRODUCTION

In the hope of exorcising some of the pathological aspects of quantum field theory and, more recently, of identifying an algebraic signature of quantum gravity, the idea of endowing spacetime with a noncommutative nature has been, over the years, actively pursued. A fairly direct approach in this direction is to consider possible deformations of the underlying kinematical Lie algebra (which codifies the nature of spacetime), led by the reasonable criterion of stability (see, *e.g.*, (7; 1)). In this paper we determine the stable form of standard quantum relativistic kinematics. We find three possible deformations, the physical implications of which depend critically on the identification of the generators sitting opposite the P's in the Heisenberg commutator — we call them Z's here, and argue against their universally accepted interpretation as position operators. Furthermore, by identifying them with the moment operators, we show that spacetime noncommutativity is not an inevitable feature of stability, contrary to previous claims.

2. LIE ALGEBRA DEFORMATIONS AND THE CONCEPT OF STABILITY

In this section we briefly review the standard Lie algebra deformation theory that will be used in the rest of the paper. Relevant references are the original source for this material (8; 9), and (7), the last one also our main motivation to follow the stability path. For background on Lie algebra and group cohomology see (4). We deal throughout with finite-dimensional real Lie algebras.

A Lie algebra $\mathcal{G} = (V, \mu)$ is constructed by providing a vector space V with a bilinear antisymmetric product $\mu : V \times V \to V$, which satisfies the Jacobi identity. Given a basis $\{T_A\}$, $A = 1, \ldots, n$ for V, the algebra $\mathcal{G}$ can be specified by its structure constants, $\mu(T_A, T_B) \equiv [T_A, T_B] = i\, f_{AB}{}^{C} T_C$. Consequently, the set $\mathcal{L}_n$ of n-dimensional real Lie algebras can be represented geometrically as a hypersurfase embedded in $\mathbb{R}^N$ (with $N = n^2(n-1)/2$) where each $f_{AB}{}^{C}$, $A < B$, runs along an axis, and where the Jacobi identities provide the defining algebraic relations[1]. The coordinates of a point P of $\mathcal{L}_n$ give the structure constants of the Lie algebra $\mathcal{G}_P$.

$GL(N, \mathbb{R})$ acts on $\mathcal{L}_n$ via linear redefinitions of the generators, $T'_A = M_A{}^B T_B$, $M \in GL(N, \mathbb{R})$, under which the structure constants transform as

$$f'_{AB}{}^{C} = M_A{}^R M_B{}^S (M^{-1})_U{}^C f_{RS}{}^U \,, \tag{1}$$

and P moves to P_M — the corresponding algebras are isomorphic. The crucial observation is that there are two types of algebras in $\mathcal{L}_n$: those that are completely surrounded by isomorphic algebras and those whose neighborhoods include non-isomorphic ones, called *stable* (or *rigid*) and *unstable*, respectively.

In physics, structure constants of Lie algebras are often identified with experimentally determined fundamental constants of a theory, *e.g.*, $\hbar$ in Heisenberg's commutator, or c^{-2} in the Lorentz algebra. The experimental errors involved render the position of the corresponding point in $\mathcal{L}_n$ uncertain. If the algebra employed is unstable, the physical predictions of the theory become ill defined, as they depend critically on the (unknown) exact values of the structure constants. If, however, the algebra is stable,

[1] $\mathcal{L}_n$ inherits the natural topology of the structure constants, *i.e.*, that of the ambient $\mathbb{R}^N$.

Proc. of the EPS-13 Conference "Beyond Einstein" – Physics for the 21st Century
11-15 July 2005, Bern, Switzerland (ESA SP-637, November 2006)

the physical theory based on it will be robust and with a wider range of validity.

Given a Lie algebra $\mathcal{G}_0 = (V, \mu_0)$, a *one-parameter (formal) deformation* of $\mathcal{G}_0$ is given by the *deformed commutator*

$$[X, Y]_t = [X, Y]_0 + \sum_{m=1}^{\infty} \psi_m(X, Y)\, t^m\,, \qquad (2)$$

where t is a formal parameter and the ψ_m are $\mathcal{G}$-valued, bilinear antisymmetric maps called *2-cochains* (over V). The definition is extended in the natural way (*i.e.*, via p-linearity and total antisymmetry) to *p-cochains* $\psi^{(p)}$, which accept p arguments[2]. The vector space of p-cochains over V will be denoted by $C^p(V)$. Notice that the 1-cochains are simply linear maps from V to V, the antisymmetry requirement being meaningless in this case. Also, 0-cochains are constant maps is V, since they accept no arguments, and may therefore be characterized by their values, making $C^0(V)$ isomorphic to V itself. Next, given an arbitrary Lie product μ, we define a *coboundary operator* s_μ, which maps p-cochains to $(p+1)$-cochains, $s_\mu\colon C^p \to C^{p+1}$, according to

$$\begin{aligned} s_\mu \triangleright \psi^{(p)}(T_{A_0}, \ldots, T_{A_p}) &= \sum_{r=0}^{p} (-1)^r \mu\left(T_{A_r},\, \psi^{(p)}(T_{A_1}, \ldots, \hat{T}_{A_r}, \ldots, T_{A_p})\right) \\ &\quad + \sum_{r<s} (-1)^{r+s} \\ &\quad \psi^{(p)}\left(\mu(T_{A_r},\, T_{A_s}),\, T_{A_0}, \ldots, \hat{T}_{A_r}, \ldots, \hat{T}_{A_s}, \ldots, T_{A_p}\right) \end{aligned} \qquad (3)$$

(hats denote omitted terms). The Jacobi identity satisfied by μ implies that s_μ is nilpotent, $s_\mu^2 = 0$. A p-cochain annihilated by s_μ will be called a *p-cocycle*, while a p-cochain in the image of s_μ, a *trivial p-cocycle* or a *p-coboundary*, the corresponding vector spaces being denoted by $Z^p(V, s_\mu)$ and $B^p(V, s_\mu)$ respectively. The p-th cohomology group of $\mathcal{G}$ is the quotient space $H^p(V, s_\mu) = Z^p(V, s_\mu)/B^p(V, s_\mu)$ in which two p-cocycles are identified if they differ by a p-coboundary.

Imposing the Jacobi identity on the deformed commutator (2) and requiring that the t-derivative, at $t = 0$, vanishes, one finds that ψ_1 in (2) satisfies $s_{\mu_0} \triangleright \psi_1 = 0$, *i.e.*, *deformations are generated by 2-cocycles*. On the other hand, effecting a linear redefinition of the generators with the matrix $M_t = I + tQ$, one finds that the $\mathcal{O}(t)$ change in μ_0 is given by the 2-cochain $s_{\mu_0} \triangleright Q$, *i.e., trivial deformations are generated by 2-coboundaries*. Geometrically, this can be interpreted as follows: the tangent space $T_{P_0}\mathcal{L}_n$ to $\mathcal{L}_n$ at P_0 is (isomorphic to) the space of integrable (see below) 2-cocycles. The subspace of $T_{P_0}\mathcal{L}_n$ leading to isomorphic Lie algebras, *i.e.*, the tangent space to the $GL(n)$-orbit Orb(P) is B^2, the space of 2-coboundaries. A sufficient condition then for the stability of $\mathcal{G}_0$ is the vanishing of its second cohomology group $H^2(\mathcal{G}_0) \equiv H^2(V, s_{\mu_0})$. In view of Whitehead's lemma, we conclude that semisimple algebras are stable.

Calculations are simplified with the introduction of the $\bar{\wedge}$ product among cochains. Put $\mathrm{Alt}^p(V) = C^{p+1}(V)$, $p \geq -1$, and $\mathrm{Alt}(V) = \bigoplus_p \mathrm{Alt}^p(V)$. Then for $\alpha \in \mathrm{Alt}^m(V)$, $\beta \in \mathrm{Alt}^n(V)$, define the product $\alpha \,\bar{\wedge}\, \beta \in \mathrm{Alt}^{m+n}(V)$ by

$$\begin{aligned} \alpha \,\bar{\wedge}\, \beta(X_0, \ldots, X_{m+n}) &= \sum_\sigma \mathrm{sgn}(\sigma) \\ \alpha\big(\beta(X_{\sigma(0)}, \ldots, X_{\sigma(n)}), X_{\sigma(n+1)}, &\ldots, X_{\sigma(m+n)}\big)\,, \end{aligned} \qquad (4)$$

where σ ranges over all permutations such that $\sigma(0) < \ldots < \sigma(n)$ and $\sigma(n+1) < \ldots < \sigma(m+n)$ (these are known as *riffle shuffles with cut at* $n+1$). The corresponding *(graded) commutator* of α, β is given by

$$[\![\alpha, \beta]\!] = \alpha \,\bar{\wedge}\, \beta - (-1)^{mn} \beta \,\bar{\wedge}\, \alpha\,. \qquad (5)$$

It is easy to see that the Jacobi identity for a 2-cochain μ takes the form $\mu \,\bar{\wedge}\, \mu = \frac{1}{2}[\![\mu, \mu]\!] = 0$ and that the action of s_μ on an arbitrary $(p+1)$-cochain $\psi \in \mathrm{Alt}^p(V)$ is given by

$$s_\mu \triangleright \psi = (-1)^p [\![\mu, \psi]\!] \equiv (-1)^p D_\mu \psi\,, \qquad (6)$$

where the second equality defines the operator $D_\mu \equiv [\![\mu, \cdot]\!]$. A useful property of D_μ is that it is a graded derivation in $\mathrm{Alt}(V)$,

$$D_\mu [\![\alpha,\, \beta]\!] = [\![D_\mu \alpha,\, \beta]\!] + (-1)^m [\![\alpha,\, D_\mu \beta]\!]\,, \qquad (7)$$

where $\alpha \in \mathrm{Alt}^m(V)$ and $\beta \in \mathrm{Alt}(V)$. Eq. (7) allows an easy derivation of the equation for *finite* deformations. If μ is a Lie product, $\mu' = \mu + \phi$ will also be one if $[\![\mu', \mu']\!] = 0$, from which one gets immediately the *deformation equation*

$$D_\mu \phi + \frac{1}{2}[\![\phi,\, \phi]\!] = 0\,, \qquad (8)$$

which reduces to the cocycle condition for infinitesimal ϕ.

Given a Lie algebra $\mathcal{G} = (V, \mu)$ and a deformation $\mu_t = \mu + \phi_t$, where $\phi_t = \sum_{n=1}^{\infty} \phi_n t^n$, substituting ϕ_t in (8) results in a series of equations for the ϕ_n, one for each power of t. The equations corresponding to t, t^2 and t^3, are

$$D_\mu \phi_1 = 0\,,\; D_\mu \phi_2 = -\frac{1}{2}[\![\phi_1,\, \phi_1]\!]\,,\; D_\mu \phi_3 = -[\![\phi_1, \phi_2]\!]\,. \qquad (9)$$

The first of (9) says that ϕ_1 is a 2-cocycle. Then the graded derivation property of D_μ implies that $[\![\phi_1, \phi_1]\!]$ is a 3-cocycle. The second of (9) may be solved for ϕ_2 provided that this 3-cocycle be a coboundary, which may not be the case if $H^3(V, D_\mu)$ is non-trivial. We conclude that the existence of non-trivial 3-cocycles may render infinitesimal deformations non-integrable. If $[\![\phi_1, \phi_1]\!]$ is indeed a trivial 3-cocycle, so that the second of (9) admits

[2]When the order p of a cochain ψ needs to emphasized, we will write $\psi^{(p)}$.

a solution, an obstruction may occur in the next step, *i.e.*, in the third of (9), and so on. It can be shown that all of these obstructions lie in H^3, so that, if H^3 is trivial, every non-trivial 2-cocycle is the first order term of some finite deformation (9). Referring back to our geometrical image of $\mathcal{L}_n$ as a hypersurface in $\mathbb{R}^N$, non-integrable 2-cocycles correspond to deformation directions that point outside of $\mathcal{L}_n$ but such that, for a little step of order t along them, the Jacobi identities are violated to order t^2, or higher.

If ϕ is a (non-trivial) 2-cocycle satisfying $[\![\phi, \phi]\!] = 0$, then Eq. (8) implies that $\mu + t\phi$, for t finite, is a Lie product, if μ is one. When the dimension of H^2 is greater than one, the vanishing of the anticommutators $[\![\phi_i, \phi_j]\!]$, $\phi_{i,j} \in H^2$, turns the finite deformation space of $\mathcal{G}$ into a vector space, since an arbitrary linear combination ϕ of the cocycles also satisfies Eq. (8). Notice that a non-trivial 2-cocycle satisfying $[\![\phi, \phi]\!] = 0$ leads to non-isomorphic algebras infinitesimally, but when extended to a finite deformation it may well lead, for particular values of t, to isomorphic algebras — we will encounter such a case in Sect. 3 below.

It is obvious from the definition given above, that a p-cochain can be realized as a $\mathcal{G}$-valued left invariant (LI) p-form on the group manifold G corresponding to $\mathcal{G}$, with the generators T_A now extended to LI vector fields. Denoting by $\{\Pi^A\}$ the LI 1-forms on G dual to the generators $\{T_B\}$,

$$\left\langle \Pi^A, T_B \right\rangle = \delta_B{}^A \,, \tag{10}$$

we write $\psi^{(p)}$ as

$$\psi^{(p)} \equiv \psi^B \otimes T_B = \frac{1}{p!} \psi_{A_1 \ldots A_p}{}^B \, \Pi^{A_1} \ldots \Pi^{A_p} \otimes T_B \,. \tag{11}$$

Then the action of s given in (18) coincides with that of an exterior covariant derivative ∇,

$$\nabla(\psi^A \otimes T_A) = (d\psi^A + \Omega^A{}_B \psi^B) \otimes T_A \,, \tag{12}$$

with the connection 1-form Ω given by

$$\Omega^A{}_B = f_{RB}{}^A \Pi^R \,, \qquad (\textit{i.e.}, \quad \nabla_{T_A} T_B = [T_A, \, T_B]) \,. \tag{13}$$

The use of the differential forms language permits writing out cochains as geometrical objects, as in (11), rather than listing their components, a practice we adhere to in the following.

3. STABLE QUANTUM RELATIVISTIC KINEMATICS

In this section we engage on the search of a stable Lie algebra containing relativisic and quantum effects. Our starting point will be the fifteen-generator algebra $\mathcal{G}_{\text{PH}}(q)$ (for "Poincaré - Heisenberg"),

$$[J_{\mu\nu}, \, J_{\rho\sigma}] = i \left(g_{\mu\sigma} J_{\nu\rho} + g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} - g_{\nu\sigma} J_{\mu\rho} \right) \tag{14}$$

$$[J_{\rho\sigma}, \, P_\mu] = i \left(g_{\mu\sigma} P_\rho - g_{\mu\rho} P_\sigma \right) \tag{15}$$

$$[J_{\rho\sigma}, \, Z_\mu] = i \left(g_{\mu\sigma} Z_\rho - g_{\mu\rho} Z_\sigma \right) \tag{16}$$

$$[P_\mu, Z_\nu] = i \, q \, g_{\mu\nu} M \,, \tag{17}$$

where $J_{\mu\nu}$ are the generators of the Lorentz group, P_μ are the momenta, Z_μ are generally identified with the positions (an interpretation we will soon challenge) and M is a central generator whose only function in the literature is to render the r.h.s. of the (covariant form of the) Heisenberg commutator, Eq. (17), linear in the generators — to our knowledge, its physical nature has never been clarified. Its omission, which is known to occur, leads to spurious non-linearities forced by the Jacobi identities. This happened in the first work to deal with non-commuting spacetime coordinates, (11), and was pointed out in (12), with the story repeating itself almost sixty years later in (5), (2), respectively. For the moment, we regard $\mathcal{G}_{\text{PH}}(q)$ as an abstract Lie algebra, devoid of any physical connotations, and inquire about its stability. Related works are (7; 6).

The 2-cochain $\mu_{\text{PH}}(q)$, corresponding to $\mathcal{G}_{\text{PH}}(q)$, is given by

$$\begin{aligned} \mu_{\text{PH}}(q) = {} & \frac{1}{2} \Pi^{\alpha\rho} \Pi_\rho{}^\beta \otimes J_{\alpha\beta} + \Pi^{\alpha\rho} \Pi_\rho \otimes P_\alpha \\ & + \Pi^{\alpha\rho} \Pi_{\dot\rho} \otimes Z_\alpha + q \Pi^\mu \Pi_{\dot\mu} \otimes M \,, \end{aligned} \tag{18}$$

where undotted indices in forms refer to P's and dotted ones to Z's, so that, *e.g.*, $\left\langle \Pi^{\dot\mu}, Z_\nu \right\rangle = \delta_\nu{}^\mu$. Slightly abusing notation, we will let Π^M denote the 1-form that detects the generator M.

We find that $H^2(\mathcal{G}_{\text{PH}}(q))$ is non-trivial (see (3) for the details of the calculations on this section),

$$H^2(\mathcal{G}_{\text{PH}}(q)) = \{[0], [\zeta_1], [\zeta_2], [\zeta_3]\} \,, \tag{19}$$

where

$$\zeta_1 = \Pi^\mu \Pi^M \otimes Z_\mu + \frac{q}{2} \Pi^\mu \Pi^\nu \otimes J_{\mu\nu} \tag{20}$$

$$\zeta_2 = -\Pi^{\dot\mu} \Pi^M \otimes P_\mu + \frac{q}{2} \Pi^{\dot\mu} \Pi^{\dot\nu} \otimes J_{\mu\nu} \tag{21}$$

$$\zeta_3 = \Pi^{\dot\mu} \Pi^M \otimes Z_\mu - \Pi^\mu \Pi^M \otimes P_\mu + q \Pi^\mu \Pi^{\dot\nu} \otimes J_{\mu\nu} \,. \tag{22}$$

We also find that all anticommutators among the ζ's vanish. Accordingly, an arbitrary linear combination $\zeta(\vec\alpha) = \alpha_i \zeta_i$ (sum over i implied), for finite α_i, provides the finite deformation $\mathcal{G}_{\text{PH}}(q, \vec\alpha)$ of $\mathcal{G}_{\text{PH}}(q)$. The deformed commutators are

$$[P_\mu, Z_\nu] = i \, q g_{\mu\nu} M + i \, q \alpha_3 J_{\mu\nu} \tag{23}$$

$$[P_\mu, P_\nu] = i \, q \alpha_1 J_{\mu\nu} \tag{24}$$

$$[Z_\mu, Z_\nu] = i \, q \alpha_2 J_{\mu\nu} \tag{25}$$

$$[P_\mu, M] = -i \, \alpha_3 P_\mu + i \, \alpha_1 Z_\mu \tag{26}$$

$$[Z_\mu, M] = -i \, \alpha_2 P_\mu + i \, \alpha_3 Z_\mu \,, \tag{27}$$

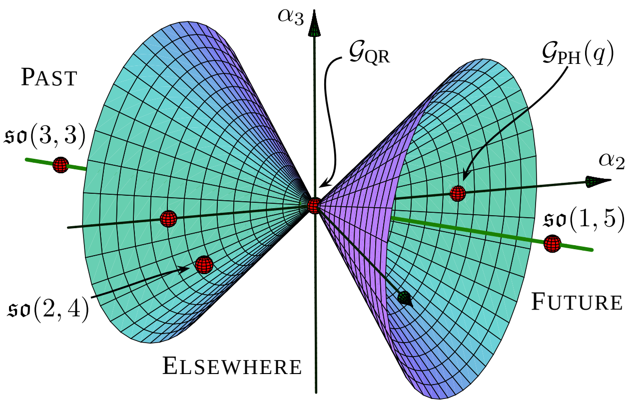

Figure 1. The $(\alpha_1, \alpha_2, \alpha_3)$ deformation space of $\mathcal{G}_{\rm PH}(q)$, with a representative of each of the six equivalence classes drawn. The two cones and the apex at the origin correspond to unstable algebras – the rest of the space to stable ones. For all classes, a representative exists with $\alpha_3 = 0$ (the little spheres denote such representatives). Standard quantum relativistic kinematics of a massive, spinless particle lies at $(0, q, 0)$, when the Z's are interpreted as moment operators.

complemented by those in Eqs. (14)–(16). For a generic deformation, the P's cease to commute among themselves, the same happens with the Z's, M is no longer central, while the Heisenberg commutator receives an additional term, proportional to $J_{\mu\nu}$.

Is $\mathcal{G}_{\rm PH}(q, \vec{\alpha})$ stable? We compute, again, the second cohomology group and find

$$H^2\big(\mathcal{G}_{\rm PH}(q,\vec{\alpha})\big) = \begin{cases} \{[0]\} & \text{if } \alpha_3^2 \neq \alpha_1\alpha_2 \\ \{[0],[\chi]\} & \text{if } \alpha_3^2 = \alpha_1\alpha_2 \end{cases}, \quad (28)$$

where $\chi = \zeta_1 + \zeta_2$ satisfies $[\![\chi, \chi]\!] = 0$. $\mathcal{G}_{\rm PH}(q, \vec{\alpha})$ is, accordingly, stable everywhere outside the instability surface $\alpha_3^2 = \alpha_1\alpha_2$ in α-space. The latter represents a double cone with the apex at the origin and its axis along the first diagonal in the α_1-α_2 plane, parallel to χ (see Fig. 1). We will refer to the various regions of α-space with their relativistic nicknames ("future", "past", *etc.*), with the positive α_1-α_2 quadrant lying in the future. It is easily shown that there are six equivalence classes of algebras, given by the regions the α-space is divided into by the double light cone: future, past, elsewhere, future cone, past cone, apex. For each of the above classes, a representative exists with $\alpha_3 = 0$. An arbitrary point in α-space may be brought on the α_1-α_2 plane by a rotation in the P_μ-Z_μ planes, $P'_\mu = \cos(\theta) P_\mu + \sin(\theta) Z_\mu$, $Z'_\mu = -\sin(\theta) P_\mu + \cos(\theta) Z_\mu$, which rotates α-space by an angle 2θ around the axis of the cone, counterclockwise as seen from the future. The α_i, $i = 1, 2, 3$, are fundamental constants of the theory of possibly Planckian and/or cosmological origin (see, *e.g.*, (5)).

As it has been pointed out in (6; 7), off the instability cone, $\mathcal{G}_{\rm PH}(q, \vec{\alpha})$ is isomorphic to some $\mathfrak{so}(m, 6-m)$, where, taking $\alpha_3 = 0$, m depends on the signs of q, α_1 and α_2. Specifically, assuming $q > 0$,

$$\mathcal{G}_{\rm PH}(q, \alpha_1, \alpha_2, \alpha_3 = 0) \cong \begin{cases} \mathfrak{so}(1,5) & \text{if } \alpha_1 > 0,\ \alpha_2 > 0 \\ \mathfrak{so}(2,4) & \text{if } \alpha_1\alpha_2 < 0 \\ \mathfrak{so}(3,3) & \text{if } \alpha_1 < 0,\ \alpha_2 < 0 \end{cases} . \quad (29)$$

4. SOME PHYSICAL CONSIDERATIONS

We deal, finally, with a number of interpretational issues. First, we wish to discuss the physical meaning of the coproduct of Lie algebra generators. We use the Poincaré algebra as an example, but the discussion applies to general Lie algebras.

Consider the state of a particle which is represented by a state vector $|\psi\rangle$ in some Hilbert space $\mathcal{H}$. To a possible transformation of the system, *e.g.*, a rotation $R_{\alpha\beta\gamma}$ parametrized by Euler's angles, one associates an operator $\mathcal{D}(R_{\alpha\beta\gamma})$, acting on $\mathcal{H}$. Imagine now that, under closer inspection, the particle is seen to be a bound state of two other particles, say, particles 1 and 2. The state space becomes $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_i$ is the state space of particle i. To rotate what is now known to be a two-particle system, one applies the above rotation to each of the constituent particle systems. This observation implies that the operator representing $R_{\alpha\beta\gamma}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is simply $\mathcal{D}_1(R_{\alpha\beta\gamma}) \otimes \mathcal{D}_2(R_{\alpha\beta\gamma})$, where $\mathcal{D}_i$ is the representation of rotations in $\mathcal{H}_i$. This is true for all representations $\mathcal{D}_i$ — we may accordingly conclude that the abstract rotation operator $R_{\alpha\beta\gamma}$ acts on tensor products as $R_{\alpha\beta\gamma} \otimes R_{\alpha\beta\gamma}$ and call this latter operator the *coproduct* $\Delta(R_{\alpha\beta\gamma})$ of $R_{\alpha\beta\gamma}$. The fact that rotations should compose in the same way, whether applied to a simple or to a composite system, is expressed algebraically by the requirement that $\Delta(R_1 R_2) = \Delta(R_1)\Delta(R_2)$. Summarizing, for all transformations S in the Poincaré group, the coproduct $\Delta(S)$ is *grouplike*,

$$\Delta(S) = S \otimes S\,, \quad (30)$$

and Δ is an algebra homomorphism,

$$\Delta(S_1 S_2) = \Delta(S_1)\Delta(S_2)\,. \quad (31)$$

Now write $S = e^A$, with A in the Poincaré algebra $\mathcal{G}_{\rm P}$ and *define* Δ to be linear in the entire $U(\mathcal{G}_{\rm P})$, the universal enveloping algebra[3] of $\mathcal{G}_{\rm P}$ — a simple calculation then shows that $\Delta(A) = A \otimes 1 + 1 \otimes A$ (this is a logarithm turning a product into a sum, as usual). We conclude that the generators of grouplike transformations are *primitive*,

$$\Delta(A) = A \otimes 1 + 1 \otimes A\,, \quad (32)$$

with $J_{\rm tot} = J_1 + J_2$ as the archetypal example from quantum mechanics. In other words, *the physical quantities corresponding to generators of grouplike transformations are additive under system composition.* All Lie algebra

[3] More precisely, a certain topological completion of $U(\mathcal{G}_{\rm P})$.

generators are of this nature. As a conclusion, only primitive operators should be allowed as Lie algebra generators.

We have seen that all the generators of the Poincaré algebra satisfy the primitiveness condition, but what about the position operators X_μ? The answer is that X_μ is not primitive. To begin with, it is rather obvious that position is not an extensive quantity: if two particles are glued together at x_μ, their composite system is also located at x_μ, not at $2x_\mu$. At the finite level, where the position operators can be regarded as generators of translations in momentum space, it is clear that translating each of the two particles forming a composite system by k in momentum space, one ends up with the composite particle being translated by $2k$, not k. Either way, it becomes evident that the position operators are not primitive and, hence, cannot be taken as generators of a Lie algebra. At the finite level, $e^{a^\mu X_\mu}$ cannot serve as points on the group manifold. This last consideration shows something about the nature of the grouplike operator that should replace e^X, the logarithm of which would be acceptable as a Lie algebra generator. Roughly speaking, it should somehow translate the particle in momentum space by a quantity proportional to its mass.

So, if X_μ is not primitive, what is its coproduct $\Delta(X_\mu)$? The answer requires to clarify what are we going to ask the position operator to do. For a single localized particle it is clear that X_μ should return its position, but what should X_μ (via its coproduct $\Delta(X_\mu)$) do on a two-particle system? Clearly, if the two particles are glued together and the composite system is localized, we should get the same answer whether we operate with X_μ on the composite particle or with $\Delta(X_\mu)$ on the two-particle system. When the two particles are far apart and/or have different velocities, the natural demand would be that $\Delta(X_\mu)$ should return the position of their center-of-momentum, which is the natural "effective position" of a relativistic composite system[4]. The problem now is that there exist various proposals for the definition of the "effective position" of a composite relativistic system, some of which imply that the latter does not behave as a 4-vector. In those cases, different observers locate the center-of-momentum of a system at different points. At the algebraic level, this means that Δ fails to be a homomorphism of the J-X commutation relations, in other words, the coproduct of the X's, in the above cases, does not even exist. For the definitions of the center-of-momentum that do yield a 4-vector, the coproduct can be read off from the expression for the center-of-momentum coordinates of a two-particle system in terms of those of the constituent particles.

In either case though, we know from experience that there are composite systems the position of which , at least approximately, behaves like a 4-vector, *e.g.*, butterflies. This implies that even though $\Delta(X_\mu)$ may not exist, depending on the definition of the center-of-momentum, it is possible to define an approximate coproduct that works on a restricted class of systems — intuitively, systems that can fool the observer into believing that they are a single, localized particle. As an example, consider the definition for the center-of-momentum of a (non-interacting) two-particle system given by

$$\vec{R} = \frac{E_1\vec{r}_1 + E_2\vec{r}_2}{E} \, , \tag{33}$$

where $E \equiv E_1 + E_2$ is the total energy of the system. Clearly, $\vec{R}$ is not part of a 4-vector. Next, consider a system such that in its center-of-momentum frame all energies E_i are nearly equal to the corresponding rest masses, $E_i \cong m_i$ — we call such systems *psychron*, from the greek word for "cold". For such a system (33) reduces to the Newtonian formula for the center-of-mass. Moreover, boosting to an arbitrary frame, all energies rescale by the same γ-factor, which cancels, so the l.h.s. does transform as a spatial vector. We conclude that, for psychron 2-particle systems, the relation

$$m_{12}x_{12}^\mu = m_1 x_1^\mu + m_2 x_2^\mu \, , \tag{34}$$

where $m_{12} \equiv m_1 + m_2$, defines the effective position x_{12} of the system as a 4-vector. So, if we denote by M the mass operator ($M^2 = P^\mu P_\mu$), we conclude from(34) that the *moment operator* $Z_\mu \equiv X_\mu M$ *is* primitive when applied to psychron systems.

To investigate the repercussions of the above interpretation of Z_μ, consider the standard quantum relativistic algebra $\mathcal{G}_{\rm QR}$, where the P's and the X's commute, and the cross-relations are given by the Heisenberg commutator. Then the Z-Z and the Z-M commutation relations are fixed, and there is no *a priori* reason why they should close linearly in the $\{J,\, P,\, Z,\, M\}$ set. Nevertheless, we find

$$[Z_\mu,\, Z_\nu] = i\, q(X_\mu P_\nu - X_\nu P_\mu)\, , \; [Z_\mu,\, M] = -i\, q P_\mu \, . \tag{35}$$

where $[X_\mu, f(P)] = -i\, q \partial f(P)/\partial P^\mu$ was used[5]. Note that the non-commutativity among the Z's is a purely quantum phenomenon ($q \neq 0$), with no connection to spacetime non-commutativity. We identify the r.h.s. of the first equation as a multiple of the orbital angular momentum operator, $L_{\mu\nu} = q^{-1}(X_\mu P_\nu - X_\nu P_\mu)$. Then, for a massive, spinless particle,

$$[Z_\mu,\, Z_\nu] = i\, q^2 J_{\mu\nu} \, . \tag{36}$$

A look at (25) and (27) shows that the above relations are of exactly the form of those provided by the α_2 deformation, with $\alpha_2 = q$. This means that $\mathcal{G}_{QR}$, the algebra of standard quantum relativistic kinematics lies at the point $(0, q, 0)$ in α-space, having only one non-trivial deformation available (along χ), which introduces non-commutativity among the momenta. The above argument shows that spacial non-commutativity, cannot of course be ruled out, but is not an inevitable feature of stabilization.

[4]See, for example, the discussion in (10), p. 84.

[5]Strictly speaking, this relation holds for functions $f(P)$ that can be expanded in power series in P — nevertheless, the commutation relations (35) are consistent with $M^2 = P^\mu P_\mu$ and we do not require anything more.

5. CONCLUDING REMARKS

We have taken in this paper the stability criterion to its ultimate consequences. Our systematic algebraic analysis has provided a general map of the possible deformations, establishing the uniqueness of previous results and shedding light on various technical issues, in particular, the interrelations among the deformations found. A fundamental departure from the standard picture has been our identification of the Z_μ generators with the moment operators of a (massive, spinless) particle, having concluded that the position operators lack the essential property of primitiveness, necessary for all Lie algebra generators.

We think that this work suggests some directions of study that deserve further consideration. First, we would like to generalize the concept of the moment operators to the case of particles with spin, and/or zero mass. Second, representation theoretical aspects of the problem should be examined, in particular, a Wigner-type classification should be carried through. It would also be of interest to develop some degree of intuition regarding the deformed kinematics, *e.g.*, by clarifying the coexistence of the Lorentz contraction with an invariant length scale.

The hope is that, after two impressive (albeit *a posteriori*) vindications of the stability criterion, in the form of the relativistic and quantum revolutions, some true predictions might await us further ahead the deformation path.

ACKNOWLEDGMENTS

C.C. would like to thank the organizers of the “Beyond Einstein: Physics for the Twentieth First Century” conference, held in Bern, Switzerland, on July 2005, for their hospitality, and partial financial support. More of the latter is also acknowledged from the DGAPA-UNAM project IN 114302 and CONACyT project 41208F.

REFERENCES

[1] C. Chryssomalakos, *Lie Superalgebra Stability and Branes*, Mod. Phys. Lett. A **16** (2001), 197–210, `hep-th/0102134`.

[2] C. Chryssomalakos and E. Okon, *Linear Form of 3-scale Relativity Algebra and the Relevance of Stability*, Int. J. Mod. Phys. D **13**, 1817 (2004), `hep-th/0407080`.

[3] C. Chryssomalakos and E. Okon, *Generalized Quantum Relativistic Kinematics: A Stability point of view*, Int. J. Mod. Phys. **13**, 2003 (2004), `hep-th/0410212`.

[4] G. Hochschild and J-P. Serre, *Cohomology of Lie Algebras*, Ann. Math. **57/3** (1953), 591–603.

[5] J. Kowalski-Glikman and L. Smolin, *Triply Special Relativity*, `hep-th/0406276`.

[6] V. V. Khruschev and A. N. Leznov, *Relativistic Invariant Lie Algebras for Kinematical Observables in Quantum Spacetime*, Grav. Cosmol. **9** (2003), 159, `hep-th/0207082`, v. 5 (March 2003).

[7] R. Vilela Mendes, *Deformations, Stable Theories and Fundamental Constants*, J. Phys. A **27** (1994), 8091–8104.

[8] A. Nijenhuis and R.W. Richardson, *Cohomology and Deformations in Graded Lie Algebras*, Bull. Amer. Math. Soc. **72** (1966), 1–29.

[9] A. Nijenhuis and R.W. Richardson, *Deformations of Lie Algebra Structures*, J. Math. and Mech. **17** (1967), 89–105.

[10] W. Rindler, *Essential relativity*, Springer-Verlag, 1979, (Revised Second Edition).

[11] H. S. Snyder, *Quantized Space-Time*, Phys. Rev. **71** (1947), 38.

[12] C. N. Yang, *On Quantized Space-Time*, Phys. Rev. **72** (1947), 874.