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Towards a First-Principles Approach to Spacetime Noncommutativity

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Abstract. Our main thesis in this note is that if spacetime noncommutativity is at all relevant in the quantum gravitational regime, there might be a canonical approach to pinning down its form. We start by emphasizing the distinction between an intrinsically noncommuting "manifold", *i.e.*, one with noncommuting coordinate functions, on the one hand, and particles with noncommuting position operators, on the other. Focusing on the latter case, which, we feel, more adequately reflects the experimental nature of our knowledge of spacetime properties, we find that several complementary considerations point to a spin-dependent noncommutativity, which is confirmed in the single-particle sector of Dirac's theory, as well as in Fokker's relativistic "center-of-mass" prescription. Finally, we propose an extension of Jordan and Mukunda's work to gain a glimpse on the effect of curvature on position operator noncommutativity.

1. Introduction

Efforts towards a quantum theory of gravity have long revolved around a noncommutative spacetime. The standard arguments in favor of such a construct, originating in quantum field theory's pathologies, are well known by now, as is the history of the idea. Previous attempts in this direction seem to follow invariably the "quantum manifold" paradigm, in which properties of classical spacetime are first captured in the (commutative) algebra of functions over it, the latter being subsequently deformed to a noncommutative "quantum" sibling. Appropriate differential calculi are then developed and, once the quantum geometry is revealed, particles and waves and other creatures are set free to roam in the quantum manifold, their motions and interactions being monitored for quantum gravitational effects. The resulting quantum spacetime candidates fall roughly in the following categories

$$\begin{aligned}
\text{Quantum spaces:} &\qquad x_\mu x_\nu = R_{\mu\nu}{}^{\rho\sigma} x_\rho x_\sigma \\
\text{Lie-type spaces:} &\qquad [x_\mu, x_\nu] = f_{\mu\nu}{}^{\rho} x_\rho \\
\text{“Canonical”:} &\qquad [x_\mu, x_\nu] = \theta_{\mu\nu} \\
\text{DSR:} &\qquad ??? = ???\,,
\end{aligned}$$

the corresponding coordinate commutation relations being given in schematic form. “Quantum”, or “q-deformed” spaces are closely related to quantum groups and typically involve quadratic relations expressed in terms of a numerical R-matrix that solves the quantum Yang-Baxter equation. In Lie-type spaces the coordinates form a Lie algebra, while in the much studied “canonical” case (a misnomer), the relevant commutator is given by a constant tensor. Finally, in “Doubly Special Relativities” (a serious misnomer), the primary arena is momentum space — when additional (Hopf algebra) structure is injected, and the situation in the dual position space is clarified, the scheme reduces to the well known κ-Minkowski spacetime[1].

A common feature in all of the above approaches is the clear separation between geometry and physics, the former fully developed before the latter can even enter the picture. This state of affairs at the formal level sharply contrasts the expected interplay of the two at the Planck scale, as exemplified by the numerous thought experiments involving ultra-energetic particles that depart from their classically expected probe (*i.e.*, spectator) behavior. The crux of the matter is that we learn about geometry by observing particles, and there seems to be nothing capturing this “experimental” aspect of geometry in any of the above schemes. What we propose instead here, is considering particles with noncommuting position operators. As we are about to see, several instances of this general idea emerge rather naturally in elementary contexts — what comes as a surprise is the persistence of a particular form of noncommutativity, involving the particle's spin.

The outline of the rest of this note is as follows: in section 2, we reiterate arguments, first put forward in [2], regarding the non-extensive nature of position operators and their subsequent inappropriateness as Lie algebra generators. We identify Lorentz boosts as the natural replacements, and go over Jordan y Mukunda's work on relativistic position operators for spinless particles [3]. It is there that we first encounter a hint for what the position operators' commutation relations might be. In section 3 we take a look at position operators in Dirac's theory, and their restrictions to the single-particle sector. We also point out the relevance of the well known (and still open) problem of the relativistic “center of mass”, and try to shed some light on the physical interpretation of noncommutativity. Section 4 outlines further work, extending that of Jordan y Mukunda, to include the effects of curvature.

2. Position operators and Lie algebras

Heisenberg's commutation relations

$$P_i X_j = -i\delta_{ij} + X_j P_i \tag{1}$$

suggest the familiar interpretation of P_i as $-i\partial/\partial X_i$. However, the same relations, read from right to left, point to the dual interpretation of X_j as $-i\partial/\partial P_j$, the latter partial acting on the left. This well known symmetry is captured in the often quoted statement that “momenta

[1] A glaring weakness of the classification outlined above is its lack of general coordinate transformation invariance — obtaining an invariant classification would be an interesting problem in itself.

translate in position space, while position operators translate in momentum space". However, as already explained in [2], the symmetry argument is flawed: translating a chair by a vector $\vec{a}$ across the room entails translating each of its halves by $\vec{a}$, while translating it by $\vec{k}$ in momentum space, entails translating each of its halves by $\vec{k}/2$. The difference lies in the fact that momentum is an extensive quantity, *i.e.*, it adds up under system composition, while position is certainly not. But to even state this, we need first to define "total momentum" and "total position" for a composite system. The former clearly refers to the sum of the momenta of the component systems, while the latter should probably refer to some sort of average position of the composite system, *i.e.*, its center-of-mass, or any of its relativistic variants. With these definitions in place, we may invoke the Newtonian expression for the center-of-mass,

$$\vec{X}_{12} = \frac{M_1 \vec{X}_1 + M_2 \vec{X}_2}{M_1 + M_2} \,, \tag{2}$$

to conclude that $K \equiv M\vec{X}$ *is* extensive, the total mass of a system being, in this limit, the sum of the masses of its components. K is nothing but the generator of (Galilean) boosts, so we learn that boosts are the extensive physical quantities associated to a particle's position, a result that we extend to the relativistic case. For reasons that we will not go into at this point, Lie algebra generators should correspond to extensive physical quantities, like momentum, angular momentum, boost, *etc.*, therefore, it is the K's that we expect to see as generators, not the X's. This is indeed the case in the Poincaré algebra,

$$\begin{array}{lll} [J_i, J_j] = \epsilon_{ijk} J_k & [J_i, K_j] = \epsilon_{ijk} K_k & [J_i, P_j] = \epsilon_{ijk} P_k \\ [K_i, K_j] = -\epsilon_{ijk} J_k & [K_i, P_j] = \delta_{ij} H & [K_i, H] = P_i \end{array}$$

where J_i, K_j, P_r and H generate rotations, boosts, space and time translations, respectively (square brackets are to be interpreted as commutators or Poisson brackets, as the need arises). Following Jordan y Mukunda [3], we look for representations appropriate for spinless particles, involving only variables $\{q_i, p_j\}$, $i, j = 1, 2, 3$, with canonical brackets $[q_i, p_j] = \delta_{ij}$, $[q_i, q_j] = 0 = [p_i, p_j]$. The answer, unique up to unitary transformations, is

$$J_i = \epsilon_{ijk} q_j p_k \qquad K_i = q_i \sqrt{\vec{p}^2 + m^2} \qquad P_i = p_i \qquad H = \sqrt{\vec{p}^2 + m^2} \,. \tag{3}$$

Next, position operators X_i, $i = 1, 2, 3$, are introduced, and the following algebraic properties are imposed on them

$$[X_i, P_j] = \delta_{ij} \qquad [J_i, X_j] = \epsilon_{ijk} X_k \qquad [X_j, K_k] = X_k [X_j, H] \,, \tag{4}$$

the last one guaranteeing that position measurements form part of a four-vector. Trying to represent X_i, one is inexorably led to $X_i = q_i$, up to unitary transformations. As a corollary, positions commute for a spinless particle. Notice that $K_i = X_i H$ in this representation. Checking the K-K commutator gives rise to three potentially non-zero terms,

$$\begin{aligned} [K_i, K_j] &= [X_i H, X_j H] \\ &= [X_i, X_j] H^2 + [X_i, H] X_j H + [H, X_j] X_i H \\ &= X_j P_i - X_i P_j \\ &= -\epsilon_{ijk} J_k \,. \end{aligned}$$

Following this computation one is led to wonder what happens when spin is present — an obvious entry in one's wishlist is

$$[X_i, X_j] = -\epsilon_{ijk} S_k / H^2 \,, \tag{5}$$

making the first term above contribute exactly the spin part to the total angular momentum[2]. Despite its dubious origins, (5) seems to capture something fundamental — we explain this statement in what follows.

3. Connections

3.1. Position operators in Dirac's theory

An obvious testing ground for (5) is Dirac's theory for spin 1/2 particles. In the standard representation, position operators act by multiplication on the wavefunctions, and are thus commutative. However, this is not a single-particle theory. The spectrum of the hamiltonian for a particle of mass m consists of the intervals $(-\infty, -m]$, $[m, \infty)$, and, in general, the various operators that appear in the theory mix the positive and negative energy subspaces, thus mapping particle to antiparticle states, and *vice-versa*. Every operator A may be decomposed into an even and an odd part, denoted by $\tilde{A}$, $\hat{A}$ respectively, such that $\tilde{A}$ preserves the sign of the energy while $\hat{A}$ flips it. When this is done to the position operators, one obtains (see [1], p. 25) $\tilde{x}_i = x_i - ...$, satisfying exactly (5), up to factors of i *etc.*, related to hermiticity and the choice of units.

3.2. The relevance of the relativistic "center of mass"

The problem of defining the relativistic analogue of the Newtonian center-of-mass for a system of particles has not been fully resolved. A good overview of the various proposals that have seen the light over the years (at least until 1948) is given by Pryce [4]. There are a number of properties that reasonable definitions should satisfy:

- **a**) The three coordinates of the "center-of-mass" should be part of a four vector, the zeroth component being the time at which they are measured.
- **b**) The "center-of-mass" should be at rest in the center-of-momentum frame.
- **c**) When no external forces act on the system of particles, its "center-of-mass" ought to move with constant velocity.
- **d**) The three coordinates of the "center-of-mass" should commute among themselves (in the sense of Poisson brackets).

The last requirement was included by Pryce with hamiltonian mechanics in mind — we, of course, are willing to drop it. The four main definitions mentioned in Pryce are

- **1**) Average of positions, weighted by rest masses. Obvious weakness: not part of a four-vector.
- **2**) Apply **1** in the center-of-momentum frame, and obtain the coordinates in any other frame by Lorentz transformation.
- **3**) Average of positions, weighted by total energies. Obvious weakness: not part of a four-vector.
- **4**) Apply **3** in the center-of-momentum frame, and obtain the coordinates in any other frame by Lorentz transformation [5].

[2] Needless to say, this is just an entry in a wishlist — there is no *a priori* guarantee that the other two terms will keep generating the orbital part of angular momentum, nor that $\vec{K} = \vec{X} H$, in the presence of spin.

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The table below summarizes the score

	a	**b**	**c**	**d**
1	-	-	-	x
2	x	-	-	x
3	-	x	x	-
4	x	x	x	-

Interestingly, the otherwise sound **3** and **4** fail to satisfy **d**, and they do so according to (5) (for **4**, H is replaced by total mass M). Another point worth further consideration is the following: in considering a system of particles, one takes their coordinates and dual momenta to satisfy the canonical Poisson bracket relations and yet, when the system is considered as one composite particle, its coordinates and dual momenta no longer satisfy the canonical relations (*e.g.*, the coordinates do not commute among themselves). The situation is rather unsatisfactory, from a physical point of view, as it necessitates knowledge of "compositeness" before the appropriate Poisson bracket algebra is selected. The problem can be further formulated in Hopf algebraic terms ("lack of coproduct") and its resolution points again to (5) as the appropriate starting point — we defer a detailed analysis along these lines to a future publication.

There are further instances where (5)-like coordinate algebras show up (*e.g.*, [6, 7]). The persistence of these relations, and the diversity of the approaches that converge to them, suggest to us that they deserve a more detailed analysis.

3.3. Why $[X, X] \sim S/H^2$*?*

We want to ponder here on what does (5) mean physically, and what it does not. To begin with, it is clear there is nothing quantum gravitational about it. Rather, the noncommutativity of the X's seems to stem from the fact that the system is not point like, but extends in space. This statement needs some refinement though. Suppose the system is observed in the center-of-momentum frame, and it is found to possess angular momentum along z (S_z *is* angular momentum in the center-of-momentum frame). Then the uncertainty relation implied by (5) specifies that the system's *center-of-mass* cannot be located *exactly* on the z-axis, but must lie within a blurry region of the x-y plane. Notice also that Newtonian systems can be extended too, and yet, their center-of-mass coordinates commute and, hence, their center-of-mass can be located exactly. This is due to the fact that, in the Newtonian limit, an extended system behaves *exactly* like a point mass located at its center-of-mass — a property not shared by relativistic systems. The uncertainty in the relativistic "center-of-mass" position then reflects exactly this absence of a sharp "mean position", and is therefore a purely relativistic effect. Its relevance for quantum gravitational considerations stems from the rather basic aspiration to find a quantum analogue to the classical geodesic motion of point particles. A quantum particle is inherently spread out, and one would like to assign to it some sort of mean position, in the hope that the latter might follow a (suitably defined) "geodesic". The most one can hope for, in view of (5), is that the effective point particle, located at the "mean position", will feel some sort of average of the metric over a region whose extend is of the order of the r.h.s. of (5).

4. The role of curvature

As mentioned already, Eqs. (5) do not capture gravitational effects, as they emerge assuming a Minkowski background. Could it be that the presence of curvature affects them? A straightforward way to address the question is to extend Jordan y Mukunda's work to de Sitter spacetime. We propose the following program:

- Write down the de Sitter algebra and introduce canonical coordinates q_i and momenta p_j, as in section 2. First, representations of the spinless case are to be sought, in the form of deformations of the ones found in [3].
- Spin variables s_i should be admitted, satisfying the rotation algebra, and representations with spin should be determined.
- Antiparticles should be introduced and the previously found representations should be appropriately extended.
- Finally, position operators X_i should be introduced and their algebraic properties imposed. This is not a trivial step, as the de Sitter momenta do not commute, and it is not clear what the appropriate Heisenberg-type relations ought to be.
- Determine a representation for the X_i and the ensuing X-X algebra.

It would suffice, for our purposes, that the computation were done perturbatively in the inverse de Sitter radius. The results would then suggest the effects expected in general spacetimes, provided the curvature changes were over an appropriate length scale.

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