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Stable Quantum Relativistic Kinematics

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ABSTRACT. We apply Lie algebra deformation theory to the problem of identifying the stable form of the quantum relativistic kinematical algebra. We find three possible deformations, introducing dimensionful invariants. It is argued that the appropriate operators to serve as Lie algebra generators are moments, not positions, leading to substantial differentiation from earlier interpretations of the nature of the deformations.

1. Introduction

There have been numerous attempts to endow spacetime with a noncommutative nature, the first hint in that direction attributed to Heisenberg. Historically, this line of thought has been pursued in the hope that some of the unpleasant aspects of quantum field theory could thus be exorcised, while more recent motivations tend to emanate from a quantum gravitational nucleus. In either case, a fairly direct approach is through the underlying kinematical Lie algebra, with the stability criterion as a sensible mathematical compass (see, *e.g.*, [**Men94, Chr01**]). In this paper we determine the stable form of standard quantum relativistic kinematics. We find three possible deformations, the physical ramifications of which rely on the identification of the generators sitting opposite the P's in the Heisenberg commutator. We call them Z's here, and argue against their universally accepted interpretation as position operators. With their proposed new role as moment operators, spacetime noncommutativity is not an inevitable feature of stability, in contrast to earlier assertions.

2. Lie Algebra Deformations and the Concept of Stability

We will be dealing with finite dimensional real Lie algebras and their deformations. We assume the reader is familiar with the relevant concepts, our brief review serving mostly to establish notation. Relevant references are [**NR66, NR67**], the original source for this material, [**Men94**], also a source and our main motivation to follow the stability path, [**Ger64**], a classic on all things deformed, and [**HS53**], for background on Lie algebra and group cohomology.

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A Lie algebra $\mathcal{G} = (V, \mu)$ is a vector space V equipped with a bilinear antisymmetric product $\mu : V \times V \to V$, which satisfies the Jacobi identity. Given a basis $\{T_A\}$, $A = 1, \ldots, n$ for V, $\mathcal{G}$ can be specified by its structure constants, $\mu(T_A, T_B) \equiv [T_A, T_B] = i\, f_{AB}{}^C T_C$. Accordingly, the set $\mathcal{L}_n$ of n-dimensional real Lie algebras is a hypersurface embedded in $\mathbb{R}^N$ (with $N = n^2(n-1)/2$) with each $f_{AB}{}^C$, $A < B$, ranging along an axis, and with the Jacobi identities as the defining algebraic relations[1]. The coordinates of a point P of $\mathcal{L}_n$ give the structure constants of the Lie algebra $\mathcal{G}_P$. $GL(N, \mathbb{R})$ acts on $\mathcal{L}_n$ via linear redefinitions of the generators, $T'_A = M_A{}^B T_B$, under which the structure constants transform as

$$f'_{AB}{}^C = M_A{}^R M_B{}^S (M^{-1})_U{}^C f_{RS}{}^U \,, \tag{2.1}$$

and P moves to P_M — the corresponding algebras are isomorphic. There exist, accordingly, two types of algebras in $\mathcal{L}_n$: those that are completely surrounded by isomorphic algebras and those that do not, called *stable* (or *rigid*) and *unstable*, respectively.

Given a Lie algebra $\mathcal{G}_0 = (V, \mu_0)$, a *one-parameter (formal) deformation* of $\mathcal{G}_0$ is given by the *deformed commutator*

$$[X,\, Y]_t = [X,\, Y]_0 + \sum_{m=1}^{\infty} \psi_m(X,\, Y)\, t^m \,, \tag{2.2}$$

where t is a formal parameter and the ψ_m are $\mathcal{G}_0$-valued, bilinear antisymmetric maps called *2-cochains* (over V). The definition is extended in the natural way (*i.e.*, via p-linearity and total antisymmetry) to *p-cochains* $\psi^{(p)}$, which accept p arguments[2]. The vector space of p-cochains over V will be denoted by $C^p(V)$. Notice that the 1-cochains are simply linear maps from V to V, the antisymmetry requirement being meaningless in this case. Also, the space of 0-cochains is V itself. *Trivial* deformations correspond to a linear redefinition of the generators with some invertible matrix, as in (2.1). Next, for a Lie algebra $\mathcal{G} = (V, \mu)$, we define a *coboundary operator* s_μ, which maps p-cochains to $(p+1)$-cochains, $s_\mu : C^p \to C^{p+1}$, according to

$$\begin{aligned} s_\mu \triangleright \psi^{(p)}(T_{A_0}, \ldots, T_{A_p}) &= \sum_{r=0}^{p} (-1)^r \mu \left(T_{A_r},\, \psi^{(p)}(T_{A_1}, \ldots, \hat{T}_{A_r}, \ldots, T_{A_p}) \right) \\ &+ \sum_{r<s} (-1)^{r+s} \psi^{(p)} \left(\mu(T_{A_r},\, T_{A_s}),\, T_{A_0}, \ldots, \hat{T}_{A_r}, \ldots, \hat{T}_{A_s}, \ldots, T_{A_p} \right) \end{aligned} \tag{2.3}$$

(hats denote omitted terms). The Jacobi identity satisfied by μ implies that s_μ is nilpotent, $s_\mu^2 = 0$. One then defines *p-cocycles* and *p-coboundaries* in the usual manner, their vector spaces being denoted by $Z^p(V, s_\mu)$ and $B^p(V, s_\mu)$ respectively. The p-th cohomology group of $\mathcal{G}$ is the quotient space $H^p(V, s_\mu) = Z^p(V, s_\mu)/B^p(V, s_\mu)$ in which two p-cocycles are identified if they differ by a p-coboundary.

Imposing the Jacobi identity on the deformed commutator (2.2) and requiring that the t-derivative, at $t = 0$, vanishes, one finds that ψ_1 in (2.2) satisfies $s_{\mu_0} \triangleright \psi_1 = 0$, *i.e.*, *deformations are generated by 2-cocycles*. On the other hand, effecting a linear redefinition of the generators with the matrix $M_t = I + tQ$, one finds that the $\mathcal{O}(t)$ change in μ_0 is given by $s_{\mu_0} \triangleright Q$, *i.e.*, *trivial deformations are generated*

[1] $\mathcal{L}_n$ inherits the natural topology of the structure constants, *i.e.*, that of the ambient $\mathbb{R}^N$.

[2] When the order p of a cochain ψ needs to emphasized, we will write $\psi^{(p)}$.

by 2-coboundaries. The geometrical picture that emerges is as follows: the tangent space $T_{P_0}\mathcal{L}_n$ to $\mathcal{L}_n$ at P_0 is (isomorphic to) Z^2, the space of 2-cocycles. The subspace of $T_{P_0}\mathcal{L}_n$ leading to isomorphic Lie algebras, *i.e.*, the tangent space to the $GL(n)$-orbit $\mathrm{Orb}(P)$ is B^2, the space of 2-coboundaries. It follows that a sufficient condition for the stability of $\mathcal{G}_0$ is the vanishing of its second cohomology group $H^2(\mathcal{G}_0) \equiv H^2(V, s_{\mu_0})$. Semisimple algebras are therefore stable, in view of Whitehead's lemma.

Calculations are simplified with the introduction of the $\bar{\wedge}$ product among cochains. Put $\mathrm{Alt}^p(V) = C^{p+1}(V)$, $p \geq -1$. Then for $\alpha \in \mathrm{Alt}^m(V)$, $\beta \in \mathrm{Alt}^n(V)$, define the product $\alpha \,\bar{\wedge}\, \beta \in \mathrm{Alt}^{m+n}(V)$ by

(2.4)
$$\alpha \,\bar{\wedge}\, \beta(X_0, \ldots, X_{m+n}) = \sum_{\sigma} \mathrm{sgn}(\sigma)\, \alpha\big(\beta(X_{\sigma(0)}, \ldots, X_{\sigma(n)}), X_{\sigma(n+1)}, \ldots, X_{\sigma(m+n)}\big)\,,$$

where σ ranges over all permutations such that $\sigma(0) < \ldots < \sigma(n)$ and $\sigma(n+1) < \ldots < \sigma(m+n)$ (these are known as *riffle shuffles with cut at* $n+1$). The corresponding *(graded) commutator* of α, β is given by

$$[\![\alpha, \beta]\!] = \alpha \,\bar{\wedge}\, \beta - (-1)^{mn} \beta \,\bar{\wedge}\, \alpha\,. \tag{2.5}$$

It may easily be shown that the Jacobi identity for a 2-cochain μ takes now the form $\mu \,\bar{\wedge}\, \mu = \frac{1}{2}[\![\mu, \mu]\!] = 0$, while the action of s_μ on an arbitrary $(p+1)$-cochain $\psi \in \mathrm{Alt}^p(V)$ is given by

$$s_\mu \triangleright \psi = (-1)^p [\![\mu, \psi]\!] \equiv (-1)^p D_\mu \psi\,, \tag{2.6}$$

where the second equality defines the operator $D_\mu \equiv [\![\mu, \cdot]\!]$. A useful property of D_μ is that it is a graded derivation in $\mathrm{Alt}(V)$,

$$D_\mu [\![\alpha,\, \beta]\!] = [\![D_\mu \alpha,\, \beta]\!] + (-1)^m [\![\alpha,\, D_\mu \beta]\!]\,, \tag{2.7}$$

where $\alpha \in \mathrm{Alt}^m(V)$ and $\beta \in \mathrm{Alt}(V)$. Graded commutators allow an easy derivation of the equation for *finite* deformations. If μ is a Lie product, $\mu' = \mu + \phi$ will also be one if $[\![\mu', \mu']\!] = 0$, from which one gets immediately the *deformation equation*

$$D_\mu \phi + \frac{1}{2}[\![\phi,\, \phi]\!] = 0\,, \tag{2.8}$$

which reduces to the cocycle condition for infinitesimal ϕ.

Given a Lie algebra $\mathcal{G} = (V, \mu)$ and a deformation $\mu_t = \mu + \phi_t$, where $\phi_t = \sum_{n=1}^{\infty} \phi_n t^n$. Substituting ϕ_t in (2.8) results in a series of equations for the ϕ_n, one for each power of t. The equations corresponding to t, t^2 and t^3, are

$$D_\mu \phi_1 = 0\,, \qquad D_\mu \phi_2 = -\frac{1}{2}[\![\phi_1,\, \phi_1]\!]\,, \qquad D_\mu \phi_3 = -[\![\phi_1, \phi_2]\!]\,. \tag{2.9}$$

The first of (2.9) says that ϕ_1 is a 2-cocycle. Then the graded derivation property of D_μ implies that $[\![\phi_1, \phi_1]\!]$ is a 3-cocycle. The second of (2.9) may be solved for ϕ_2 provided that this 3-cocycle be a coboundary, which may not be the case if $H^3(V, D_\mu)$ is non-trivial. We conclude that the existence of non-trivial 3-cocycles may render infinitesimal deformations non-integrable. If $[\![\phi_1, \phi_1]\!]$ is indeed a trivial 3-cocycle, so that the second of (2.9) admits a solution, an obstruction may occur in the next step, *i.e.*, in the third of (2.9), and so on. It can be shown that all of these obstructions lie in H^3, so that, if H^3 is trivial, every non-trivial 2-cocycle is the first order term of some finite deformation [**NR67**]. Referring back to our geometrical image of $\mathcal{L}_n$ as a hypersurface in $\mathbb{R}^N$, non-integrable 2-cocycles correspond to deformation directions that point outside of $\mathcal{L}_n$ but such that, for a little step of order t along them, the Jacobi identities are violated to order t^2, or higher. Concrete examples of stable Lie algebras with non-trivial second cohomology group have been constructed, see, *e.g.*, [**Ric67**].

If ϕ is a (non-trivial) *nilpotent* 2-cocycle, *i.e.*, satisfying $[\![\phi, \phi]\!] = 0$, then Eq. (2.8) implies that $\mu + t\phi$, for t finite, is a (non-isomorphic) Lie product, if μ is one. When the dimension of H^p is greater than one, the vanishing of the anticommutators $[\![\phi_i, \phi_j]\!]$, $\phi_i \in H^p$, turns the finite deformation space of $\mathcal{G}$ into a vector space, since then an arbitrary linear combination ϕ of the cocycles satisfies Eq. (2.8). Notice that a nilpotent non-trivial 2-cocycle leads to non-isomorphic algebras infinitesimally, but when extended to a finite deformation as above it may well lead, for particular values of t, to isomorphic algebras — we will encounter such a case in Sect. 3 below.

It is clear from the definition given above that a p-cochain can be realized as a Lie algebra-valued p-form on the corresponding group manifold — we often make use of this fact in what follows.

3. Stable Quantum Relativistic Kinematics

Consider the fifteen-generator algebra $\mathcal{G}_{\rm PH}(q)$ (for "Poincaré - Heisenberg"),

$$[J_{\mu\nu},\, J_{\rho\sigma}] = i\left(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}\right) \tag{3.1}$$

$$[J_{\rho\sigma},\, P_\mu] = i\left(g_{\mu\sigma}P_\rho - g_{\mu\rho}P_\sigma\right) \tag{3.2}$$

$$[J_{\rho\sigma},\, Z_\mu] = i\left(g_{\mu\sigma}Z_\rho - g_{\mu\rho}Z_\sigma\right) \tag{3.3}$$

$$[P_\mu, Z_\nu] = i\, q\, g_{\mu\nu} M\,, \tag{3.4}$$

all other commutators being zero. $J_{\mu\nu}$ are the generators of the Lorentz group ($J_{\mu\nu} = -J_{\nu\mu}$), P_μ are the momentum 4-vector components, Z_μ is generally identified with the position 4-vector components (an interpretation we will soon challenge) and M is a central generator whose only function in the literature is to render the r.h.s. of the (covariant form of the) Heisenberg commutator, Eq. (3.4), linear in the generators — to our knowledge, its physical nature has never been clarified. Its omission, which is known to occur, leads to spurious non-linearities forced by the Jacobi identities. This happened in the first work to deal with non-commuting spacetime coordinates, Ref. [**Sny47**], and was pointed out in [**Yan47**], with the story repeating itself almost sixty years later in [**KGS04**], [**CO04**], respectively. For the moment, we regard $\mathcal{G}_{\rm PH}(q)$ as an abstract Lie algebra, devoid of any physical connotations, and inquire about its stability. Related works are [**Men94, KL03**].

The 2-cochain $\mu_{\rm PH}(q)$, corresponding to $\mathcal{G}_{\rm PH}(q)$, is given by

$$\mu_{\rm PH}(q) = \frac{1}{2}\Pi^{\alpha\rho}\Pi_\rho{}^\beta \otimes J_{\alpha\beta} + \Pi^{\alpha\rho}\Pi_\rho \otimes P_\alpha + \Pi^{\alpha\rho}\Pi_{\dot\rho} \otimes Z_\alpha + q\Pi^\mu\Pi_{\dot\mu} \otimes M\,, \tag{3.5}$$

where undotted indices in forms refer to P's and dotted ones to Z's, so that, *e.g.*, $\langle \Pi^{\dot\mu}, Z_\nu\rangle = \delta_\nu{}^\mu$. Slightly abusing notation, we will let Π^M denote the 1-form that detects the generator M.

We find that $H^2(\mathcal{G}_{\rm PH}(q))$ is non-trivial,

$$H^2(\mathcal{G}_{\rm PH}(q)) = \{[0], [\zeta_1], [\zeta_2], [\zeta_3]\}\,, \tag{3.6}$$

where

$$\zeta_1 = \Pi^{\mu}\Pi^{M} \otimes Z_\mu + \frac{q}{2}\Pi^{\mu}\Pi^{\nu} \otimes J_{\mu\nu} \tag{3.7}$$

$$\zeta_2 = -\Pi^{\dot\mu}\Pi^{M} \otimes P_\mu + \frac{q}{2}\Pi^{\dot\mu}\Pi^{\dot\nu} \otimes J_{\mu\nu} \tag{3.8}$$

$$\zeta_3 = \Pi^{\dot\mu}\Pi^{M} \otimes Z_\mu - \Pi^{\mu}\Pi^{M} \otimes P_\mu + q\Pi^{\mu}\Pi^{\dot\nu} \otimes J_{\mu\nu}\,. \tag{3.9}$$

We also find that all anticommutators among the ζ's vanish. Accordingly, an arbitrary linear combination $\zeta(\vec\alpha) = \alpha_i\zeta_i$ (sum over i implied), for finite α_i, provides the finite deformation $\mathcal{G}_{\rm PH}(q,\vec\alpha)$ of $\mathcal{G}_{\rm PH}(q)$. The deformed commutators are

$$[P_\mu, Z_\nu] = i\,q g_{\mu\nu} M + i\,q\alpha_3 J_{\mu\nu} \tag{3.10}$$

$$[P_\mu, P_\nu] = i\,q\alpha_1 J_{\mu\nu} \tag{3.11}$$

$$[Z_\mu, Z_\nu] = i\,q\alpha_2 J_{\mu\nu} \tag{3.12}$$

$$[P_\mu, M] = -i\,\alpha_3 P_\mu + i\,\alpha_1 Z_\mu \tag{3.13}$$

$$[Z_\mu, M] = -i\,\alpha_2 P_\mu + i\,\alpha_3 Z_\mu\,, \tag{3.14}$$

to be supplemented by Eqs. (3.1)–(3.3). For a generic deformation, the P's cease to commute among themselves, the same happens with the Z's, M is no longer central, while the Heisenberg commutator receives an additional term, proportional to $J_{\mu\nu}$.

Is $\mathcal{G}_{\rm PH}(q,\vec\alpha)$ stable? We compute, again, the second cohomology group and find

$$H^2\big(\mathcal{G}_{\rm PH}(q,\vec\alpha)\big) = \begin{cases} \{[0]\} & \text{if } \alpha_3^2 \neq \alpha_1\alpha_2 \\ \{[0],[\chi]\} & \text{if } \alpha_3^2 = \alpha_1\alpha_2 \end{cases}, \tag{3.15}$$

where $\chi = \zeta_1 + \zeta_2$ satisfies $[\![\chi,\chi]\!] = 0$. $\mathcal{G}_{\rm PH}(q,\vec\alpha)$ is, accordingly, stable everywhere outside the instability surface $\alpha_3^2 = \alpha_1\alpha_2$ in α-space. The latter represents a double cone with the apex at the origin and its axis along the first diagonal in the α_1-α_2 plane, parallel to χ (see Fig. 1). We succumb to the temptation to refer to the various regions of α-space with their relativistic nicknames ("future", "past", *etc.*), with the positive α_1-α_2 quadrant lying in the future. It is easily shown that there are six equivalence classes of algebras, given by the regions the α-space is divided into by the double light cone: future, past, elsewhere, future cone, past cone, apex. For each of the above classes, a representative exists with $\alpha_3 = 0$. An arbitrary point in α-space may be brought on the α_1-α_2 plane by a rotation in the P_μ-Z_μ planes, $P'_\mu = \cos(\theta)P_\mu + \sin(\theta)Z_\mu$, $Z'_\mu = -\sin(\theta)P_\mu + \cos(\theta)Z_\mu$, which rotates α-space by an angle 2θ around the axis of the cone, counterclockwise as seen from the future. The same kind of rotation may be used to bring points in the future and past cones and the elsewhere, on the α_1-α_3 or the α_2-α_3 plane. In any case, the α_i, $i = 1,2,3$, are fundamental constants of the theory of possibly Planckian and/or cosmological origin (see, *e.g.*, [**KGS04**]).

As it has been pointed out in [**KL03, Men94**], off the instability cone, $\mathcal{G}_{\rm PH}(q,\vec\alpha)$ is isomorphic to some $\mathfrak{so}(m, 6-m)$, where, taking $\alpha_3 = 0$, m depends on the signs of q, α_1 and α_2. Specifically, assuming $q > 0$,

$$\mathcal{G}_{\rm PH}(q,\alpha_1,\alpha_2,\alpha_3=0) \cong \begin{cases} \mathfrak{so}(1,5) & \text{if } \alpha_1 > 0,\ \alpha_2 > 0 \\ \mathfrak{so}(2,4) & \text{if } \alpha_1\alpha_2 < 0 \\ \mathfrak{so}(3,3) & \text{if } \alpha_1 < 0,\ \alpha_2 < 0 \end{cases}. \tag{3.16}$$

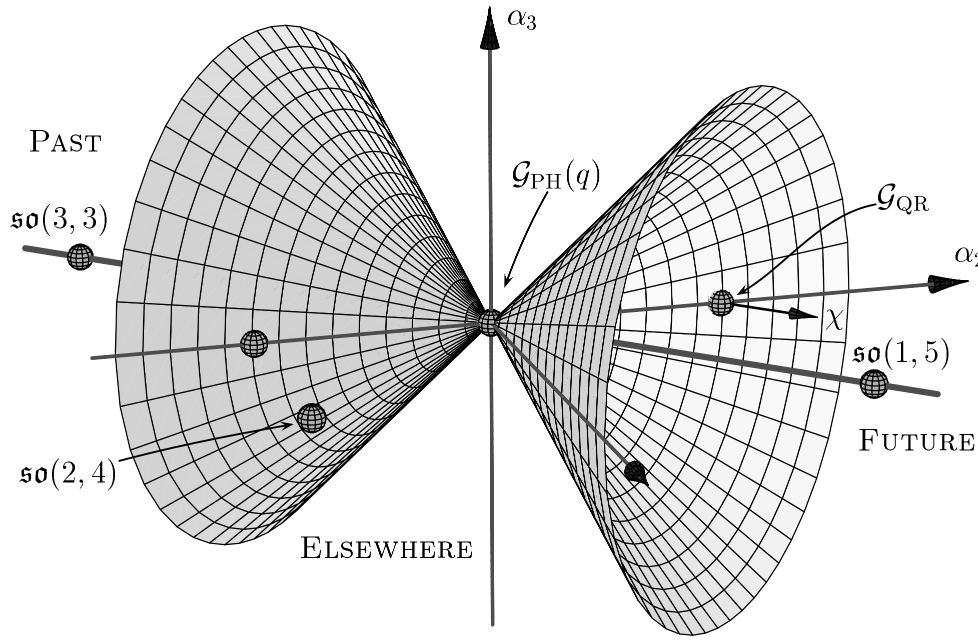

Figure 1. The $(\alpha_1, \alpha_2, \alpha_3)$ deformation space of $\mathcal{G}_{\mathrm{PH}}(q)$, with a representative of each of the six equivalence classes drawn. The two cones and the apex at the origin correspond to unstable algebras – the rest of the space to stable ones. For all classes, a representative exists with $\alpha_3 = 0$ (the little spheres denote such representatives). Standard quantum relativistic kinematics of a massive, spinless particle lies at $(0, q, 0)$, when the Z's are interpreted as moment operators.

4. Some Physical Considerations

Consider the group manifold $G_{\mathrm{PH}}(q, \vec{\alpha})$, corresponding to the algebra $\mathcal{G}_{\mathrm{PH}}(q, \vec{\alpha})$. A point g on it represents some operation, *e.g.*, a rotation, and, sufficiently close to the identity, one may write $g = e^A$, for some A in $\mathcal{G}_{\mathrm{PH}}$. Consider now arbitrary functions f, h on $G_{\mathrm{PH}}(q, \vec{\alpha})$ — pointwise multiplication says that $(fh)(g) = f(g)h(g)$, implying commutativity among functions. The dual *coproduct* is given by $\Delta(g) = g \otimes g$, making g *group-like*, so that

$$\langle fh, g\rangle = \langle f \otimes h,\, \Delta(g)\rangle = \langle f \otimes h,\, g \otimes g\rangle = \langle f,\, g\rangle\langle h,\, g\rangle \equiv f(g)h(g)\,. \tag{4.1}$$

For the generator A of g this implies $\Delta(A) = A \otimes 1 + 1 \otimes A$, making A *primitive.* The conclusion is that Lie algebra generators are primitive operators.

Apart from being dual to the product among functions, the coproduct dictates the way an operator acts on tensor products of representations. For example, the group-like coproduct for the rotation g above implies that, to rotate a two-particle system, one must rotate, by the same rotation, each of the constituent particles. At the infinitesimal level, this results in the Leibniz-like rule $J_{\mathrm{tot}} = J_1 + J_2$, familiar from the quantum theory of angular momentum. Thus, primitiveness of the generators translates into additivity of the corresponding physical quantities under system composition.

One may substitute boosts instead of rotations in the example above, or translations, thus exhausting the Poincaré algebra generators. But do the *position* operators, call them X_μ, share this property? The answer is clearly no. At the infinitesimal level, it is obvious that position is not additive under system composition. At the finite level, where the position operators can be regarded as generators of translations in momentum space, it is clear that translating each of the two particles forming a composite system by k in momentum space, one ends up with the composite particle being translated by $2k$, not k. Either way, it becomes evident that the position operators are not primitive and, hence, cannot be taken as generators of a Lie algebra. At the finite level, $e^{a^\mu X_\mu}$ cannot serve as points on the group manifold.

But there is more. Acting on a two-particle system, via its coproduct, the position operator should, presumably, return the center-of-momentum position for the system. The latter is not a 4-vector, if Eq. (4.2) below is taken as its definition, so that different inertial observers identify it with different points in spacetime (see, *e.g.*, [**Rin79**], p. 84). At the algebraic level, this means that Δ fails to be a homomorphism of the J-X commutation relations, in other words, the coproduct of the X's does not, in general, exist. On the other hand, we know from experience that the position of certain composite systems does behave like a 4-vector, at least approximately (bullets come to mind). What is common in those systems is that they are sufficiently lumped together to fool the observer into perceiving them as a single, localized particle. Trying to formalize this a bit, we notice that the expression for the center-of-momentum position of a (non-interacting) two-particle system,

$$\vec{R} = \frac{E_1 \vec{r}_1 + E_2 \vec{r}_2}{E}\,, \tag{4.2}$$

with $E = E_1 + E_2$, does define, approximately, the spatial part of a 4-vector when the energies E_i are approximately equal to the rest masses in the center-of-momentum frame. We call such systems *psychron*, from the Greek ψυχρόν for "cold". In that case, boosting to an arbitrary frame, all energies rescale by the same γ-factor, which cancels, and the l.h.s. of (4.2), given now by the Newtonian expression for the center-of-mass, transforms as a vector. Multiplying both sides of that equation by M (the mass operator), and ignoring ordering ambiguities, we find that the *moment* operator $Z_\mu = X_\mu M$ *is* primitive, and *does* behave like a 4-vector, when applied to psychron systems. We are inclined therefore, as our notation might have given already away, to identify the Z's in our stability analysis with the moment operators introduced here. Notice that doing so, the Heisenberg commutator $[P_\mu, X_\nu] = i\,q g_{\mu\nu}$ becomes Eq. (3.4), with the so far cryptic M finally revealed to be the mass operator.

Our good fortune does not stop here. Consider the standard quantum relativistic algebra $\mathcal{G}_{\mathrm{QR}}$, with commuting P's, commuting X's, and the Heisenberg commutator among them. Then the Z-Z commutation relations are fixed, and there is no *a priori* reason why they should close linearly in the $\{J,\, P,\, Z,\, M\}$ set. Nevertheless, we find

$$[Z_\mu,\, Z_\nu] = i\,q(X_\mu P_\nu - X_\nu P_\mu)\,, \qquad [Z_\mu,\, M] = -i\,q P_\mu\,. \tag{4.3}$$

We recognize the r.h.s. of the first equation as (a multiple of) the covariant form of the orbital angular momentum operator, $L_{\mu\nu} = q^{-1}(X_\mu P_\nu - X_\nu P_\mu)$. For a massive, spinless particle then,

$$[Z_\mu,\, Z_\nu] = i\, q^2 J_{\mu\nu}\,. \tag{4.4}$$

A glance at (3.12), (3.14), shows that the above commutation relations are of exactly the form found earlier, with $\alpha_2 = q$ and $\alpha_1 = \alpha_3 = 0$. This places $\mathcal{G}_{\mathrm{QR}}$ at the point $(0, q, 0)$ in α-space, with the only available non-trivial deformation (along χ) introducing non-commutativity among the momenta. This last observation may not rule out the compatibility of noncommuting spacetime coordinates with the deformation found above but it does show that it is not an inevitable feature of stability.

5. Concluding Remarks

We only have space to mention some directions for future work.

- Elucidating the nature of the moment operators in the general case, including massless and/or spinful (*sic*) particles would be desirable.
- A Wigner-like classification, and other representation issues, should be clarified.
- The possibility that among the deformations found, some are compatible with non-commuting spacetime coordinates should be examined.
- Phenomenological implications of the deformations should be analyzed.

The hope is that, after two spectacular *a posteriori* vindications of the stability point of view, in the form of the relativistic and quantum revolutions of the last century, some true predictions might await us further ahead the sinuous deformation path.

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