Position Operators and Center of Mass: New Perspectives

P. Aguilar, C. Chryssomalakos, H. Hernandez Coronado,

Instituto de Ciencias Nucleares
Circuito Exterior s/n, Ciudad Universitaria
Universidad Nacional Autónoma de México
CP 04510 México, D.F., MÉXICO
pedro.aguilar,chryss,hcoronado@nucleares.unam.mx

E. Okon

Instituto de Investigaciones Filosóficas
Circuito Mario de la Cueva s/n, Ciudad Universitaria
Universidad Nacional Autónoma de México
CP 04510 México, D.F., MÉXICO
eokon@filosoficas.unam.mx

Abstract: After reviewing the work of Pryce on Center-of-Mass (CoM) definitions in special relativity, and that of Jordan and Mukunda on position operators for relativistic particles with spin, we propose two new criteria for a CoM candidate: associativity, and compatibility with the Poisson bracket structure. We find that they are not satisfied by all of Pryce's definitions, and they also rule out Dixon's CoM generalization to the curved spacetime case. We also emphasize that the various components of the CoM position do not commute among themselves, in the general case, and thus provide a natural entry point to the arena of noncommutative spacetime, without the *ad-hoc* assumptions of the standard paradigm.

Contents

1 Introduction

We begin by stating our main motivation in examining in detail the concept of a position operator: classic works advocating spacetime noncommutativity invoke the inescapable gravitational disturbance caused by ultra-energetic probes as its conceptual origin. We read, for example, in [9]:

> Our proposal differs radically: attempts to localize with extreme precision cause gravitational collapse so that spacetime below the Planck scale has no operational meaning. We elaborate on this well known remark and are led to spacetime uncertainty relations.

Yet, the standard paradigm of noncommutative spacetime physics starts off considering an intrinsically noncommuting manifold, *i.e.*, one where the coordinate functions are promoted to elements of a noncommutative algebra, builds upon it an analogue of a differential calculus, and only then, long after the noncommutative structure has fully crystalized, are particles and fields allowed to storm in, and their properties to be studied. Thus, spacetime noncommutativity is separated drastically, and paradoxically, from the very entities supposedly responsible for it. Furthermore, if a particle's presence is assumed to disturb spacetime to the point that an effective noncommutative geometry emerges, then it would seem natural that different particles, with different properties, could perceive different effective geometries, and there is no room for such geometric pluralism in the standard paradigm. Finally, envisioning a physicist's approach to geometry, one where points and curves and, indeed, all geometrical concepts, are given meaning through *gedanken* experiments, involving realistic particles and fields, one is led to focus on the position operators of the probes, or their CoM[1], if they are extended, as the only means of extracting spatial information about spacetime, given that the abstract noncommuting coordinates seem ill-suited to operational considerations.

The subject of position operators is by no means new, although our particular focus on noncommutativity might have an element of novelty. Early discussions arose with the advent of special relativity [16, 10, 31], culminating in the work of Newton and Wigner [28], and a comprehensive review by Pryce [32], that we present in some detail later on. At about the same time attempts were made at generalizations of the special relativistic proposals to curved spacetimes [30], which in subsequent years reached maturity in the work of Dixon [5, 6, 7, 8] and Beiglböck [2]. A second wave of scrutiny arrived in the sixties and seventies [34, 11, 12, 13, 29, 20, 23, 24], particularly in regards to the incompatibility of relativistic quantum mechanics and the concept of a localized particle, with fresh insights registering as late as in the nineties [25, 14, 21]. An excellent, and relatively recent, survey is reference [15]. In what regards the particular connection with noncommutativity, reference [26] pointed out the relevance of algebraic stability considerations, a theme that was later taken up, in more detail, in [4]. Despite a long, refined, and instructive history, the subject has made only modest incursions in standard textbooks, Schweber's [33] and Greiner's [19] being among the most detailed treatments.

Section 2 serves the dual purpose of summing up the work of Pryce [32] and Jordan and Mukunda [22], on the one hand, while interspersing comments and marking departures from these references in our view of the matter. Section 3 proposes two new criteria for a CoM recipe that we consider fundamental — strangely, they do not seem to appear in any reference that we know of. The reader will also find in this section a discussion of some finer points, that we found worthy of a comment. The paper ends with a summary of our findings, and some open questions.

2 CoM and Position Operators

The core of this section is a review of various CoM definitions (section 2.1), based mostly on the work of Pryce [32], and an overview of Jordan and Mukunda's approach to relativistic position operators of particles with spin [22] (section 2.2). Some differences in our understanding of the subject matter are also pointed out.

[1]We use the term Center-of-Mass (CoM) as if in quotation marks, meaning some sort of average position, not necessarily the newtonian one.

2.1 CoM in Special Relativity

We start by summarizing the overview given by Pryce in 1948 [32] of the various proposals for a special relativistic version of the newtonian CoM position vector, commenting, along the way, on conceptual differences and refinements that have emerged in the intervening sixty five years. Pryce starts by listing a set of desirable properties for its components, and then evaluates various existing proposals against this list. Thus, in a perfect world, according to Pryce, the following should all be true[2] (see also [3]):

1) The three spatial coordinates of the CoM should be part of a four vector, the zeroth component being the time at which they are measured.

2) The CoM should be at rest in the center-of-momentum frame.

3) When no external forces act on the system of particles, its CoM ought to move with constant velocity.

4) The three coordinates of the CoM should commute among themselves (in the sense of Poisson brackets).

Next come the various candidates that have been proposed over the years:

a) The good old newtonian recipe, also endorsed by Eddington [10]: average of positions, weighted by rest masses. Weakness: not part of a four-vector.

b) Apply **a** in the center-of-momentum frame, and obtain the coordinates in any other frame by Lorentz transformation.

c) Average of positions, weighted by total energies, also known as *centroid* — studied in detail by Fokker [16]. Weakness: not part of a four- vector.

d) Apply **c** in the center-of-momentum frame, and obtain the coordinates in any other frame by Lorentz transformation — Fokker calls this the *invariant mass-centre* [16].

e) This is a strange one, with no particular claims to elegance: average of recipes **c** and **d** above, weighted by total energy and total rest mass, respectively (see (3) below). Introduced by Pryce in [31], studied also by Newton and Wigner in [28]. Despite doubtful aesthetics, we will have more to say about it later on.

Pryce carefully evaluates these candidates — the results are summarized in the table below

	1	**2**	**3**	**4**
a	-	-	-	✓
b	✓	-	-	✓
c	-	✓	✓	-
d	✓	✓	✓	-
e	-	✓	✓	✓

[2]Whether these requirements are independent is worth discussing, but our aim is just a reasonably complete list, not axiomatics.

As can be appreciated, our world is not perfect, at least not in the sense of Pryce. Regarding recipes **a** and **b**, and despite their distinct newtonian flavor, the special relativistic context underlying our discussion dictates that the particle mass that appears in their definition satisfy $m^2 = p^2$, p being the corresponding four-momentum. In principle, the above verbal description, as given by Pryce, is ambiguous, as one can still contemplate at least two options for, say, recipe **a**, applied to a two-point-particle system:

$$X_I^{\mathbf{a}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\,, \qquad X_{II}^{\mathbf{a}} = \frac{m_1 x_1 + m_2 x_2}{\sqrt{(p_1 + p_2)^2}}\,, \tag{1}$$

where, in the second option, the rest mass of the composite object appears in the denominator. Despite it being the natural choice, from a special relativistic point of view, this second option suffers from the rather serious shortcoming of locating the CoM of two momentarily coincident particles, with different velocities, at a point distinct from their common position. This, we feel, leaves only the first option, and points to a general feature of every reasonable CoM definition, assuming that, when applied to a point particle system, it involves a weighted sum over the positions of the particles: the sum of the weights must equal unity.

Applied to a general inertial reference frame, the first of (1) gives the 3D position vector (X, Y, Z) of the CoM, pointing to the intersection A of an instantaneous, say, $t = T$ 3-plane, with the particle's worldline. Boosting to a second frame is accompanied by a change in the orientation of the instantaneous (in the second frame) 3-planes. The time T' the position measurement is effected in the second frame is chosen so that the "inclined" $t' = T'$ 3-plane intersects the particle's worldline also at A. Criterion **1** above requires that (T', X', Y', Z') be related to (T, X, Y, Z) via the Lorentz transformation matrix corresponding to the boost connecting the two frames. That **a** fails this criterion is easily shown by examining the following situation: refering to figure 1, consider two identical particles on the x-axis, approaching the origin Q with opposite velocities. At $t = 0$, the particles are at A and B, respectively, with $\overline{AQ} = \overline{BQ}$, so that their worldines meet along the t-axis, at R. The equality of the masses implies that the CoM's worldline is QR (since Q is the midpoint of AB), *i.e.*, the t-axis itself. In a second frame, moving to the right, the $t' = 0$ line is inclined, and the two particles, at that instant (in the second frame) are at A' and B' respectively. Again, the equality of their masses, which still holds in the moving frame, dictates that the CoM's worldline in the moving frame be the line $Q'R$, where Q' is the midpoint of $A'B'$ (both worldlines must pass through R, where the two particles coincide). Clearly, the two worldlines are distinct — they are not even parallel. Recipe **b**, on the other hand, satisfies **1** by construction. Both **a** and **b** fail, in general, properties **2** and **3** because they cannot be generalized so as to include the fields that mediate the possible interactions between the point particles. As a result the CoM wiggles around, as the momentum carried by these fields is unaccounted for. Finally, assuming the standard Poisson bracket relations for the coordinates and momenta of the individual particles that make up the composite system, the CoM of which we are interested in, it is easy to show that the various components of the CoM position vector defined by **a** and **b** commute among themselves, and satisfy those same standard relations

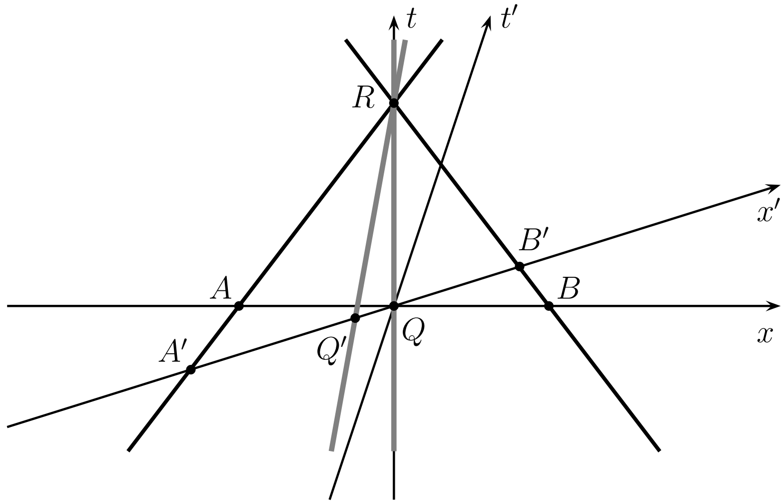

Figure 1: Spacetime diagram of a two-particle system for which recipe **a** fails the Lorentz covariance criterion **1**. The CoM worldlines measured in the two frames are shown in grey.

with the total momentum of the system.

Regarding Lorentz-covariance, the discussion for recipes **c** and **d** is analogous to the one above, with **c** replacing **a**, and **d** replacing **b**. The non-covariance of **c** is made intuitive by a standard argument, illustrated in figure 2. A rotating disk is observed in its center-of-momentum frame, where its centroid is clearly at its geometrical center. In a frame where the disk moves to the right, the increase in the energy density of its upper half, compared to that of the lower half, shifts the centroid along the positive vertical axis — the centroid worldlines observed in the two frames are parallel to each other. Corresponding quantitative results can be found in [32], [27] (p. 170-173) and [15] (section 8). Recipes **c** and **d**, on the other hand, relying on energy rather than rest mass, *can* be extended to include the contribution of the fields, and thus score positively on columns **2** and **3**. But there is a price to pay. The total energy of the individual particles depends on all components of their momentum, so that, *e.g.*, the x-coordinate of the CoM depends on the y-component of the momentum of the i-th particle, p^i_y, while the y-coordinate of the CoM depends, naturally, on y^i. It is no surprise then that the Poisson bracket of the x and y-coordinates of the CoM is nonzero, in general, for recipes **c** and **d**. Calling $\mathbf{q}$ the CoM position according to **c**, and $\mathbf{X}$ the one according to **d**, one computes the Poisson brackets

$$\{q_1, q_2\} = -\frac{S_3}{E^2}\,, \qquad \{X_1, X_2\} = -\frac{S_3}{M^2}\,, \tag{2}$$

where $\mathbf{S}$ is the total spin of the system[3], and E, M, its total energy and rest mass, respectively. This failure of commutativity is remedied in recipe **e** by a judicious choice

[3]Spin is given by $\mathbf{S} = \mathbf{J} - \mathbf{q} \times \mathbf{P}$, where $\mathbf{J}$, $\mathbf{P}$ denote the total angular momentum and momentum, respectively, of the system.

Figure 2: A rotating disk, in its center-of-momentum frame (left), and moving as a whole to the right (right — schematically drawn, no relativistic effects shown). The centroid, in each case, is marked by a dot.

of the weights used in averaging over **c** and **d**,

$$\tilde{\mathbf{q}} = \frac{E\mathbf{q} + M\mathbf{X}}{E + M} \,, \tag{3}$$

$\tilde{\mathbf{q}}$ denoting the CoM position according to **e**. Back in 1948, Pryce was understandably worried about the -'s in the intersection of rows **c**, **d** with column **4** of the table above, as they signaled trouble with the standard Poisson structure of hamiltonian mechanics. Today, we can be more relaxed about it, and, even, thankful, realizing that standard relativistic mechanics supplies the seeds for a noncommutative, from the operational point of view, spacetime. Furthermore, as has been pointed out in [3], this particular form of noncommutativity seems to appear, with slight variations, in many different contexts, and might be worthy of a deeper analysis.

A final comment is due regarding property **1**. In short, we believe its importance is overrated. The CoM is not a physical point, that can scratch the laboratory walls or poke a hole through a screen. Rather, it is a mathematical point where the extended object may be mentally collapsed, retaining some of its effects in the surroundings. Consider for example a uniform sphere floating in front of observer A. The gravitational field it produces at A's position will not change if the entire mass of the sphere is collapsed to its center. At a different location in the room, observer B, whose frame is related to that of A, *e.g.*, by a rotation, also identifies the center of the sphere as its CoM, and measures therefore CoM coordinates related to those measured by A by a rotation matrix — we conclude that this particular recipe for the CoM is covariant under rotations. But this covariance depends crucially on the exact r^{-2} law for the gravitational field. If this law is modified to $r^{-2+\epsilon}$, the CoM for A, still defined as the point where if the entire mass is collapsed, the gravitational field at A's position will be left unchanged, will move along the radial line defined by A, and will not coincide any more with that of B. In an analogous manner, the worldline of an extended relativistic object's CoM simply marks the trajectory of an "equivalent" point particle, with some freedom being available what the equivalence is exactly based on. Thus, already at the qualitative level of the above discussion, it emerges that there is no necessity in requiring the Lorentz covariance of a CoM's worldline with the same urgency, for example, that this is done for real point particles. Having said that, it is clear that **1** would be a highly convenient, from a practical point of view, feature of a CoM recipe.

2.2 Position Operators in Relativistic Quantum Mechanics

We turn now to the related concept of a position operator for a relativistic point particle. The relation to the CoM discussion is obvious, in one direction: a CoM recipe applied to a point particle ought to provide a position operator for that particle. The converse question is not as trivial: given a prescription for a position operator, applicable to point particles, is there any canonical way to apply it to an extended object, transmuting the position operator recipe to a CoM one? We will have something to say about this latter question in section 3. What we will do at this point, will be to first give a summary of the beautiful work of Jordan and Mukunda, reported in [22], following it by a discussion of some of its ramifications. The subject of [22] is a Lorentz covariant position operator for relativistic point particles with spin. The basic idea is to define such an operator algebraically, through its commutators (or Poisson brackets) with the generators of the Poincaré group, and then look for representations of these relations. Three cases are studied in succession: positive energy spinless particles, positive energy particles with spin, positive and negative energy particles with spin.

One begins by postulating the existence of ten infinitesimal generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, for time translations, space translations, rotations, and Lorentz boosts, respectively, satisfying the Poincaré algebra

$$\begin{array}{lll} [P_i, P_j]\,, = 0 & [P_i, H] = 0\,, & [J_k, H] = 0\,, \\ [J_i, P_j] = \epsilon_{ijk} P_k\,, & [J_i, J_j] = \epsilon_{ijk} J_k\,, & [J_i, K_j] = \epsilon_{ijk} K_k \\ [K_i, H] = P_i\,, & [K_i, P_j] = \delta_{ij} H & [K_i, K_j] = -\epsilon_{ijk} J_k\,, \end{array} \tag{4}$$

where the brackets denote Poisson brackets in classical mechanics, and commutators divided by i in quantum mechanics. Then position operators for point particles are introduced, and are postulated to satisfy the relations

$$[x_j, P_k] = \delta_{jk} \qquad [J_i, x_j] = \epsilon_{ijk} x_k \qquad [x_j, K_k] = \frac{1}{2} \left(x_k [x_j, H] + [x_j, H] x_k \right)\,, \tag{5}$$

the third of which owes its apparent complexity to the fact that, under a Lorentz boost, the simultaneity hypersurface of an observer changes, and with it, its intersection with the particle worldline, which defines the particle's position for that observer. The above relations capture the desired geometrical behavior of a position operator, under the symmetry transformations of the underlying spacetime. Additionally to the above, and conceptually distinct, is the requirement that the components of $\mathbf{x}$ commute among themselves,

$$[x_i, x_j] = 0\,, \tag{6}$$

which the authors of [22] also impose, with motivation similar to that of Pryce in the previous subsection.

2.2.1 Positive energy spinless particles

The case of a particle of positive mass m and zero spin, for which Pryce's recipes **c**, **d** and **e** all coincide, is examined first. Hermitean operators are sought which satisfy (4) and generate the positive energy irreducible unitary representation of the Poincaré group labeled by mass m and zero spin. The representation space $\mathcal{H}$ is generated by hermitean coordinates $\mathbf{q}$ and partials $\mathbf{p}$ satisfying the canonical relations

$$[q_i, q_j] = 0\,, \qquad [p_i, p_k] = 0\,, \qquad [q_i, p_j] = \delta_{ij}\,. \tag{7}$$

An essentially (*i.e.*, up to unitary equivalence) unique solution is found, the canonical form of which is

$$H^0 = \sqrt{\mathbf{p}^2 + m^2} \equiv W\,, \tag{8}$$

$$\mathbf{P}^0 = \mathbf{p}\,, \tag{9}$$

$$\mathbf{J}^0 = \mathbf{q} \times \mathbf{p}\,, \tag{10}$$

$$\mathbf{K}^0 = \frac{1}{2}(W\mathbf{q} + \mathbf{q}W)\,. \tag{11}$$

Now solutions are sought of the relations (5) — the answer again is unique:

$$\mathbf{x}^0 = \mathbf{q}\,, \tag{12}$$

implying $[x^0_i, x^0_j] = 0$ as a corollary, so that all conditions imposed on the position operator are satisfied. The Newton-Wigner position operator [28] also reduces to this form, for a spinless particle, although the fact is hidden by the difference in the Hilbert space measure used in [28].

2.2.2 Positive energy particles with spin

In this case the particle has spin s different from zero, and the representation space is accordingly augmented to $\mathcal{H}_{\text{spin}} \otimes \mathcal{H}$ by the introduction of a hermitean operator $\mathbf{S}$, with

$$[S_i, q_j] = 0\,, \qquad [S_i, p_j] = 0\,, \qquad [S_i, S_j] = \epsilon_{ijk} S_k\,. \tag{13}$$

The Poincaré generators are now represented irreducibly as

$$H^s = W\,, \tag{14}$$

$$\mathbf{P}^s = \mathbf{p}\,, \tag{15}$$

$$\mathbf{J}^s = \mathbf{q} \times \mathbf{p} + \mathbf{S}\,, \tag{16}$$

$$\mathbf{K}^s = \frac{1}{2}(W\mathbf{q} + \mathbf{q}W) + \frac{\mathbf{p} \times \mathbf{S}}{W + m}\,. \tag{17}$$

The only generators to suffer a change, compared to the previous case, are **J** and **K**, and the extra term added to the latter is worth an interpretation. As it has been mentioned before, the above representations can be unitarily conjugated, $A \to A' = UAU^{-1}$, with $U^\dagger = U^{-1}$ and A any of the above generators, giving rise to new representations. Each of the representations thus obtained corresponds to a different prescription for measuring the physical quantities involved. For example, given a particle with spin, and an arbitrary inertial observer (frame), a prescription must be given regarding which axis the spin is measured along. The above representation corresponds to the following prescription: first boost your frame to the particle's rest frame, and then measure the spin along the z-axis of the boosted frame. When two observers are present, the frames of which differ by an infinitesimal boost, and each of them boosts to the particle's rest frame, the resulting boosted frames differ by an infinitesimal rotation, even though the original frames were parallel to each other. It is exactly this rotation that is generated, in spin space, by the second term in the expression for $\mathbf{K}^s$ above. Indeed, an infinitesimal boost with rapidity $\boldsymbol{\eta}$ is generated by $\boldsymbol{\eta} \cdot \mathbf{K}^s$, the second term of which, when substituting (17), is $(W+m)^{-1}\mathbf{p}\times\mathbf{S}\cdot\boldsymbol{\eta} = (W+m)^{-1}\boldsymbol{\eta}\times\mathbf{p}\cdot\mathbf{S}$, the last form showing that the rotation generated in spin space is $(W+m)^{-1}\boldsymbol{\eta}\times\mathbf{p}$, in accordance with the one dictated by composing boosts according to the last of (4).

One inquires now about hermitean solutions to (5), and the essentially unique answer turns out to be

$$\mathbf{x}^s = \mathbf{q} - a\frac{(\mathbf{p}\cdot\mathbf{S})\mathbf{p}}{W(W+m)} + a\mathbf{S} - \frac{\mathbf{p}\times\mathbf{S}}{m(W+m)}\,, \tag{18}$$

where a is an arbitrary real number[4]. Puting $a = 0$ in the above expression one recovers the CoM recipe $\mathbf{d}$ of Pryce, applied to point particles. On the other hand, there is no real value of a for which the components of $\mathbf{x}^s$ commute — a pure imaginary solution for a exists, which, however, renders $\mathbf{x}^s$ non-hermitean. Summarizing, for positive energy particles with spin, a one-parameter family of hermitean operators exists, all of which satisfy (5), but none of which satisfies (6).

2.2.3 Positive and negative energy particles with spin

Finally, we consider particles with spin that have access to both positive and negative energy states. The representation space is further augmented to $\mathcal{H}_E \otimes \mathcal{H}_{\text{spin}} \otimes \mathcal{H}$ and a new hermitean operator $\boldsymbol{\rho}$ is introduced, satisfying

$$[\rho_i, q_j] = 0\,, \qquad [\rho_i, p_j] = 0\,, \qquad [\rho_i, S_j] = 0\,, \qquad [\rho_i, \rho_j] = 2\epsilon_{ijk}\rho_k\,. \tag{19}$$

The essentially unique representation of the Poincaré algebra is, in this case,

$$H^E = \rho_3 W\,, \tag{20}$$

$$\mathbf{P}^E = \mathbf{p}\,, \tag{21}$$

$$\mathbf{J}^E = \mathbf{q}\times\mathbf{p} + \mathbf{S}\,, \tag{22}$$

$$\mathbf{K}^E = \frac{1}{2}\rho_3(W\mathbf{q} + \mathbf{q}W) + \rho_3(W+m)^{-1}\mathbf{p}\times\mathbf{S}\,, \tag{23}$$

[4]There are some remarks concerning the validity of (18) that the interested reader can consult in [22], right before Eq. (3.7).

amounting to a sign insertion in H^s and $\mathbf{K}^s$ for the negative energy states (we assume $\rho_i = \sigma_i$, with σ_i the Pauli matrices), so as to obtain a direct sum of two irreducible representations, with the same mass and spin, but opposite energies. Looking for hermitean solutions to (5) one finds

$$\mathbf{x}^E = \mathbf{q} - \rho_2 \frac{(\mathbf{p}\cdot\mathbf{S})\mathbf{p}}{W^2(W+m)} + \rho_2 \frac{\mathbf{S}}{W} + \frac{\mathbf{p}\times\mathbf{S}}{W(W+m)} \,, \tag{24}$$

which seems even less inspiring than the one in (18). Nevertheless, things conspire to give $[x_i^E, x_j^E] = 0$, *i.e.*, commutativity is miraculously restored. The above $\mathbf{x}^E$ is the only solution of (5) that reduces to $\mathbf{x}^E = \mathbf{q}$ when $\mathbf{S} = 0$ — a more detailed discussion of the uniqueness of this solution is given in [22].

The fact that $\mathbf{x}^E$ is represented by a non-local pseudodifferential operator means that $\mathbf{q}$-space is not physical position space in this case (similar remarks hold true for $\mathbf{x}^s$). One can try to switch to physical space by a unitary transformation, $\mathbf{x}^E \to \tilde{\mathbf{x}}^E = U\mathbf{x}^E U^{-1}$, such that $\tilde{\mathbf{x}}^E = \mathbf{q}$ — there is a chance that this might be possible since the components of $\mathbf{x}^E$ commute among themselves. One can further restrict the transformation by requiring that it leave invariant the canonical forms of $\mathbf{P}$ and $\mathbf{J}$ in (21), (22). The unique unitary operator U that implements this transformation is given by $U = e^{iV}$, with

$$V = -\rho_2 p^{-1}(\mathbf{p}\cdot\mathbf{S}) \arctan\left(\frac{p}{m}\right) \,, \tag{25}$$

where $p^2 = \mathbf{p}\cdot\mathbf{p}$, *i.e.*,

$$e^{iV}\mathbf{x}^E\, e^{-iV} = \mathbf{q} \,, \qquad e^{iV}\mathbf{p}\, e^{-iV} = \mathbf{p} \,, \qquad e^{iV}\left(\mathbf{q}\times\mathbf{p}+\mathbf{S}\right) e^{-iV} = \mathbf{q}\times\mathbf{p}+\mathbf{S} \,. \tag{26}$$

It is of some interest to find out what is the form the hamiltonian in (20) acquires after the U-transformation. Straightforward manipulations result in

$$\tilde{H}^E = e^{iV} H^E e^{-iV} = \rho_3 m + 2\rho_1\, \mathbf{p}\cdot\mathbf{S} \,. \tag{27}$$

But $2\mathbf{S}$, $\boldsymbol{\rho}$ are just two mutually commuting copies of the Pauli matrices $\boldsymbol{\sigma}$, *i.e.*,

$$2S_i = \mathbf{1}\otimes\sigma_i \,, \qquad \rho_i = \sigma_i \otimes \mathbf{1} \,, \tag{28}$$

where $\mathbf{1}$ denotes the unit 2 by 2 matrix, so that

$$\rho_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \beta \,, \qquad 2\rho_1\mathbf{S} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} = \boldsymbol{\alpha} \,, \tag{29}$$

and

$$\tilde{H}^E = \beta m + \boldsymbol{\alpha}\cdot\mathbf{p} \tag{30}$$

is just the Dirac hamiltonian. Thus, as has already been emphasized in [22], the sequence of steps taken in this section may be considered as an alternate derivation of the Dirac equation, based on the requirement that a Lorentz-covariant position operator for relativistic point particles with spin exist. Additionally, the failure of x_i^s to commute among themselves, and the intriguing reinstatement of commutativity in the case of $\mathbf{x}^E$ highlight from a novel point of view the intricate interrelationship between the availability of negative energy states and localization.

3 Further Developments

We present now some additional results regarding position operators and CoM recipes that complement what has been presented in the previous section. The considerations presented here arose during our investigations of possible generalizations of the special relativistic position operators and CoM recipes discussed above to curved spacetimes. There is of course a long bibliography on these matters, ranging from the early work of Papapetrou [30] to Dixon's [5, 6, 7, 8] and Beiglböck's [2] subsequent refinements. It was therefore surprising for us to discover that some of what we consider to be basic aspects of the problem have been meticulously ignored over the years. We propose, accordingly, two additional criteria for prospective CoM recipes, *associativity* and *canonical algebra homomorphism* (CAH), the meaning of which we clarify in what follows.

3.1 CoM associativity

It will be convenient, in order to explain what we mean by associativity, to begin with the simplest possible (but non-trivial) application of a CoM prescription: to find the CoM of two point particles, P_A and P_B. Roughly speaking, we are looking for an "equivalent" particle, call it P_{AB}, which can replace the pair, in a certain prescribed sense. Given the worldlines and masses of the two particles, a CoM prescription ought to specify the worldline *and mass* of P_{AB}. We formalize this concept in the following way: all the information about a (classical) point particle is contained in its associated energy momentum tensor, which has support on its worldline. Thus, a CoM prescription defines a product $*$ between such tensors, so that if P_A, P_B are described by tensors T_A, T_B, then their CoM is described by $T_{AB} = T_A * T_B$.

A property of paramount importance in the practical applications of the Newtonian CoM is the associatitvity of the corresponding $*$-product. Thus, to compute the CoM of three objects, P_A, P_B, and P_C, one can, and very often does, compute first the CoM of P_A, P_B, replaces the pair by the equivalent object P_{AB}, and then computes the CoM of the pair P_{AB}, P_C. The result turns out the same if one proceeds the other way around, first combining P_B with P_C, and then the result, P_{BC}, with P_A, which translates into the associativity condition for the $*$-product of the corresponding energy-momentum tensors. We advocate that this is a sensible and most useful property to ask for, and elevate it to property **5**, extending Pryce's list. In the absence of this property, if a physicist, after years of effort and hard work, manages to calculate the CoM of the universe, and then a fly comes by, which somehow had escaped his attention, then to include its contribution in the total CoM the entire caclulation has to be repeated from scratch: a non-associative CoM recipe applies only to the entire object, there is no modularity in its calculation. An associative CoM $*$-product guarantees that the two-point-particle CoM prescription is sufficient to define the CoM of any extended object, including any relevant fields. We emphasize that, in our view, a CoM recipe not only defines an effective worldline for an extended object, but also specifies an effective point particle, following that worldline — that is why our $*$-product maps to point particle energy-momentum tensors, not just time-like curves.

3.2 Atoms *vs.* molecules, or, the quest for CAH

Our second addition to Pryce's wish list has to do with an unsatisfactory feature of various of the standard CoM candidates that, we feel, is even less acceptable than the lack of associativity. The problem we perceive is the following: in studying a composite object, one starts by assuming the standard commutation relations among the positions and momenta of each of the various particles involved[5], *i.e.*,

$$[x^i, x^j] = 0\,, \qquad [p_i, p_j] = 0\,, \qquad [x^i, p_j] = i\delta^i{}_j\,, \tag{31}$$

with commutators of quantities referring to different particles all vanishing. Then one computes the coordinates X^i of the CoM, and its associated total momentum P_j, both as functions of the x's and the p's of all the particles, and then checks whether the CoM quantities X^i, P_j satisfy the same commutation relations as those assumed for the constituent particles, Eq. (31). The answer is, often, negative, as is the case, *e.g.*, for definitions **c** and **d** of Pryce. This state of affairs implies that, essentially, we have one set of rules for elementary particles, and a different one for composite ones. Thus, when presented with a new, unknown particle, the theorist must inquire about its *ultimate* inner structure before being able to decide which set of formulas to employ in its description. This aspect of a CoM prescription we find disturbing, so we propose to amend Pryce's list so as to discourage its proliferation. We will say that a CoM prescription, together with a particular algebra structure among the dynamical variables (*e.g.*, coordinates and momenta), defines a CAH, if the algebra of the X^i and P_j is identical to that of the x^i and p_j of individual elementary particles, the latter not necessarily being the canonical one of (31). Note that this is a condition on a *pair* of data: a CoM prescription, taken together with a particular algebra structure among the dynamical variables — it enters Pryce's extended list under number **6**.

To formalize our requirement, we introduce the *canonical algebra* $\mathcal{F}$ of functions of the single-particle operators x^i, p_j — just what functions we admit in $\mathcal{F}$ we will not attempt to specify at this point, contending ourselves with the minimalist requirement that the algebra among them, inferred from that of the basic variables, be well defined. For a two-particle system then the appropriate function algebra is $\mathcal{F}\otimes\mathcal{F}$, to which the product of $\mathcal{F}$, denoted by simple concatenation of the factors, extends as

$$(a\otimes b)(c\otimes d) = ac\otimes bd\,. \tag{32}$$

A map $\mathcal{D}\colon \mathcal{F}\to\mathcal{F}\otimes\mathcal{F}$ is a homomorphism of $\mathcal{F}$ if $\mathcal{D}(ab)=\mathcal{D}(a)\mathcal{D}(b)$. On the other hand, applying a particular CoM prescription to a two-particle system, we determine functions X^i, P_j of the two particles' data (positions, momenta, mass, spin, *etc.*), that naturally

[5] In this section, square brackets stand for commutators.

live in $\mathcal{F} \otimes \mathcal{F}$. For example, the newtonian CoM prescription gives rise to

$$X^i = \frac{m_1 x_1^i + m_2 x_2^i}{m_1 + m_2} \to \frac{M x^i \otimes 1 + 1 \otimes M x^i}{M \otimes 1 + 1 \otimes M}, \tag{33}$$

$$P_i = p_1 + p_2 \to p_i \otimes 1 + 1 \otimes p_i, \tag{34}$$

where M is the mass operator of the extended galilean algebra, assumed to commute with x^i and p_j. We may now state our requirement as follows: a CoM prescription will be said to define a CAH, if, when applied to a two-particle system, defines functions X^i, P_j, as in the newtonian example above, such that the map

$$\mathcal{D}\colon \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}, \qquad x^i \mapsto X^i, \quad p_i \mapsto P_i, \tag{35}$$

is a homomorphism of the canonical algebra $\mathcal{F}$. Assuming a particular CoM prescription defines a CAH $\mathcal{D}$, we may also capture the associativity property mentioned above by requiring that $\mathcal{D}$ be *coassociative*,

$$(\mathcal{D} \otimes \mathrm{id}) \circ \mathcal{D} = (\mathrm{id} \otimes \mathcal{D}) \circ \mathcal{D}, \tag{36}$$

i.e., if $\mathcal{D}$ is applied twice, it shouldn't matter which tensor factor it is applied to the second time ("id" in the above expression denotes the identity map). Thus, in summary, our two additions to Pryce's list, entries **5** and **6**, can be compactly expressed as the requirement that *the CoM prescription should define a coassociative homomorphism of* $\mathcal{F}$. In our formulation so far, there is no unique identity element for the $*$-product, which amounts to saying that, technically, $\mathcal{F}$ is a bialgebra without counit — we are currently working on remedying this.

3.3 Living in the right algebra

Motivated by an example from the Dirac theory of spin 1/2 particles, we propose in this subsection an algebraic criterion for a position operator, which achieves the following:

1. It provides the transmutation, alluded to earlier, of a position operator definition to a CoM one.

2. It guarantees that the above CoM prescription defines a CAH.

We refer back to section 2.2.3, and Eqs. (20)–(23), where the representation of the Poincaré algebra, appropriate for particles with both positive and negative energy states and spin, was given. We refer to this as the *energy representation* because the hamiltonian is in block-diagonal form. This guarantees that positive energy spinors have their lower two components equal to zero, and *vice-versa* for the negative energy ones. A glance at the above equations shows that all generators are represented by even operators, *i.e.*, operators that, just like the hamiltonian, do not mix states with energies of opposite sign. This is not so for the corresponding position operator $\mathbf{x}^E$, the eveness of which is spoiled by the presence of ρ_2. The consequences are dear, and lead to pathologies, like *zitterbewegung* and a velocity operator with only eigenvalues ± 1. At the same time, in this representation $\mathbf{x}^E$ assumes a rather uninspiring non-local form, signaling that the physical interpretation of the corresponding $\mathbf{q}$-space is nontrivial — in particular, $\mathbf{q}$-space is not physical position space. It was long ago noticed in [17][6] that the physical quantity represented by $\mathbf{q}$ has the rather suggestive time-derivative $[\mathbf{q}, H^E] = \rho_3 \mathbf{p}/W$, which is just the usual relativistic expression for the velocity of a particle, allowing for negative energy states, without any trembling, called in [17] *mean velocity*. Working backwards, one recognizes $\mathbf{q}$ as representing mean-position $\bar{\mathbf{X}}$, *i.e.*, $\bar{\mathbf{X}}^E = \mathbf{q}$. $\bar{\mathbf{X}}$ is the position-like quantity through which the Dirac theory acquires a smooth non-relativistic limit. But its virtues are not limited to this: unlike $\mathbf{x}^E$, it is even, and can be expressed in terms of the Poincaré generators. Indeed, a few dull lines of algebra show that

$$\bar{\mathbf{X}}^E = \left((\mathbf{K}\cdot\mathbf{P})\frac{\mathbf{P}}{HP^2} - (\mathbf{J}\times\mathbf{P})P^{-2} - \left(\mathbf{K} + i\frac{\mathbf{P}}{2H}\right)\frac{H+m}{P^2} - i\frac{\mathbf{P}}{2H^2}\right)\left(1 - \frac{H(H+m)}{P^2}\right)^{-1}, \tag{37}$$

where $m \equiv (H^2 - \mathbf{P}^2)^{1/2}$. As we are about to show, this fact has important implications.

Physical quantities that appear as Lie algebra generators are (usually) extensive, *i.e.*, additive under system composition — this is the case, for example, in the Poincaré Lie algebra, considered for concreteness in the quantum context, *i.e.*, with brackets in (4) proportional to commutators. This fact is captured algebraically in that the *coproduct* map

$$\Delta\colon A \mapsto \Delta(A) = A\otimes 1 + 1\otimes A\,, \tag{38}$$

where A is any generator, is a homomorphism of the Lie algebra,

$$\Delta([A,B]) = [\Delta(A), \Delta(B)]\ , \tag{39}$$

as can be easily checked (B, above, denotes also a generator of the algebra). Further, if Δ is extended as a homomorphism to the entire universal enveloping algebra (UEA) of the Lie algebra, it will respect the commutator structure of *functions* of the generators. Thus, defining $\Delta(AB) = \Delta(A)\Delta(B)$, with A, B generators of the Lie algebra, and similarly for higher order products, we are guaranteed that

$$[\Delta(F), \Delta(G)] = \Delta([F,G])\,, \tag{40}$$

where now F and G are functions of the generators[7]. These simple facts suffice to prove that any position operator that can be expressed in terms of the generators of some underlying symmetry Lie algebra (*e.g.*, $\bar{\mathbf{X}}^E$ in (37)), satisfies the two properties enumerated at

[6] Two remarks are appropriate here: first, the authors of [17] actually choose to work in the Dirac representation, which is certainly not as convenient, for the particular calculation, as the one we work in here. Second, the mean position operator we are about to introduce, was discussed even earlier, by several authors — see the discussion in [34] and references therein.

[7] Strictly speaking, one can only admit monomials of the generators to start with, but more general functions can be considered by working in an appropriate completion of the universal enveloping algebra.

the beginning of this subsection. Summarizing, a CoM prescription that, when applied to a single particle, gives rise to a position operator in the UEA of an underlying symmetry Lie algebra ("property **7**"), satisfies automatically properties **5** and **6** above. It is worth mentioning that the mean position operator, represented by $\bar{\mathbf{X}}^E$ in (37), that enjoys, as we explained above, such distinguished properties, coincides with the CoM prescription **e** of Pryce, applied to a single particle (see also footnote 7 in [17]).

3.4 Discussion

We collect here a few remarks regarding criteria **5** (associativity), **6** (CAH), and **7** ($\in$ UEA). Of the three, the first two we consider fundamental — despite this we have not been able to locate any reference in the literature where they are even mentioned.

Criterion **5** seems innocuous, but both **a** and **b** actually fail it, since the denominator in their definitions contains the sum of the masses, while the composite object mass is given by the relativistic $m_{12}^2 = (p_1 + p_2)^2$. Apart from this, in our study of possible generalizations of Pryce's recipes to curved spacetimes, we have found that many of our attempts, and those of others before us, fail it. In particular, Dixon's construction [5], that reduces to Pryce's recipe **d** in the flat spacetime limit, and is widely considered the last word on the subject, is non-associative, a pathology that, to our knowledge, has not been pointed out before.

Criterion **6** requires special care in its implementation – we illustrate the subtleties involved with a particular example. Recipe **d** of Pryce gives rise to a CoM position operator $\mathbf{X}$, the components of which satisfy the second of (2), while the coordinates and momenta of the constituent particles are assumed to satisfy the standard Poisson bracket relations. Thus, one is inclined to conclude that the pair (**d**, canonical PB structure) fails **6**. On the other hand, Pryce ([32], equation (2.11)) gives the following expression for $X^\mu = (X^0, \mathbf{X})$ at $t = 0$ (with $m^2 = P^2$)

$$X^\mu = \frac{J^{\mu\nu} P_\nu}{m^2} - \frac{J^{0\nu} P_\nu P_\mu}{m^2 P^0} , \tag{41}$$

implying that **d** satisfies our criterion **7**, which, it was argued above, is sufficient for both **5** and **6** to hold. The resolution of this apparent contradiction lies in that if (41) is applied to the constituent particles, then the various components of an individual particle position operator will fail to commute by a term proportional to the particle's spin, which Pryce simply assumes to be zero, while composite systems made of spinless particles may well have nonzero total spin. The proper implementation of **6** then gives rise to the following statement: if the second of (2), together with the corresponding X-P and P-P relations, is termed "Pryce d PB structure", then the pair (**d**, Pryce d PB structure) satisfies **6**. In other words, the Pryce d PB structure is stable under system composition, when the composite system position is given by **d** — we find this a remarkable property, sufficient to single out a particular CoM recipe. But then the pair (**e**, canonical PB structure) also satisfies **6**, so there are still forks down the road to *the* CoM recipe. An appropriate extension of **6** might be criterion $\bar{\mathbf{6}}$, which requires that *there exist* a PB structure for the constituent particles, such that the CoM recipe in question reproduce that same PB structure for the composite object quantities. The drawback of this formulation is that it is in general hard to prove that a recipe fails $\bar{\mathbf{6}}$. Proving that it satisfies it is considerably easier. In fact, *any* associative CoM recipe satisfies $\bar{\mathbf{6}}$. To see this, consider an associative recipe, say, **h**, applied to a composite system S, and call hPB the Poisson bracket structure of S (viewed as a single composite particle), and iPB the one assumed for the constituent particles. If iPB coincides with hPB, then **h** satisfies $\bar{\mathbf{6}}$ by definition. If the two PB structures are different, imagine dividing S in two subsystems, S_1 and S_2. If **h** is associative, the CoM of S can be calculated by first calculating the CoM's of S_1, S_2, and then combining these two to find that of S. But the coordinates and momenta of S_1, S_2 satisfy hPB, and by combining them we know we recover again hPB. So the pair (**h**, hPB) satisfies **6**, and, hence, **h** satisfies $\bar{\mathbf{6}}$. There are of course a number of assumptions underlying the argument, the most critical being that fPB has a fixed form, for any number (greater than 1) of particles. This will be the case if hPB can be expressed in terms of the values of the ten Poincaré generators for the composite system. But then the only reasonable choice for iPB is to use that same form to derive hPB for individual particles, *i.e.*, one should had started by assuming hPB for the constituent particles.

If the above fPB involves nontrivial X-X brackets, then upon quantization, one ends up with a quantum theory with noncommuting position components. In particular, for **d**, one gets a position uncertainty for elementary particles with spin, in the plane normal to the spin, that is of the order of their Compton wavelength. A similar result holds in the classical case: a classical relativistic system with spin, has a minimal radius in the plane normal to the spin [27] (p. 173), a result related to the noncovariance of the centroid, and crucially dependent on a positive energy density condition. This collection of results could give rise to a questioning of the point-like nature of elementary particles with spin, an idea the authors of [15], for example, toy with, but we feel the suggestion is insufficiently motivated, as the little miracle of section 2.2.3, in which negative energy states restore commutativity, seems to imply that nature has found a way to circumvent positivity-based theorems.

Finally, we emphasize that criterion **7** involves expressing the position operator as member of (a suitable extension of) the UEA of the symmetry Lie algebra — this should not be confused with fortuitous such expressions, valid only in a particular irreducible representation. Thus, m^2 in (37) stands indeed for the quadratic Casimir of the Poincaré algebra, not a multiple of the identity operator. Accordingly, our discussion above only applies to systems of massive particles, since $P^2 = m^2$ appears in the denominator in (37).

4 Summary

After reviewing Pryce's review of CoM recipes, and Jordan and Mukunda's approach to position operators for relativistic particles with spin, we proposed two new criteria for a CoM candidate, namely, that it ought to be associative, and reproduce the chosen canonical algebra of the dynamical variables, the latter criterion refering to the pair (CoM recipe, canonical algebra). We also showed that if the CoM can be expressed in terms of the generators of the underlying symmetry algebra (Poincaré, in our case), then both criteria mentioned above are satisfied. The situation is summarized in the table that follows

	1	**2**	**3**	**4**	**5**	$\bar{\mathbf{6}}$	**7**
a	-	-	-	✓	-	CCR	-
b	✓	-	-	✓	-	CCR	-
c	-	✓	✓	-	✓	2.1	✓
d	✓	✓	✓	-	✓	2.2	✓
e	-	✓	✓	✓	✓	CCR	✓

The first four columns just repeat the table shown in section 2. Column **5** shows that the two "newtonian" CoM definitions fail associativity. Column $\bar{\mathbf{6}}$ reveals that all recipes reproduce *some* canonical structure: **a**, **b**, and **e** the standard one (CCR), **c** the one in the first of (2), and **d** the second of them. Column **7** remarks that **c**, **d** and **e** are all expressible in terms of the Poincaré generators, a virtue not shared by **a** and **b**.

We expect our two new criteria, **5** and **6**, to be even more relevant in the curved spacetime case, and plan on pursuing this matter in the future. Some preliminary work has shown that the question of CAH involves impenetrable algebra, even in concrete, simple cases, like that of de Sitter spacetime. Associativity is easier to deal with, especially if it fails, as one can work perturbatively in the ratio of the object's size to the de Sitter radius, or some other small parameter. Such investigations tie in nicely with related explorations of the effective spacetime geometry perceived when realistic clocks and meter sticks and extended probes are used (see, for example, [18, 1] and references therein), all these efforts aiming at elucidating alternative aspects of quantum gravity.

Acknowledgements

We would like to acknowledge partial financial support from DGAPA-UNAM projects IN114712 (CC), IA400312 (EO), and CONACyT project 103486 (CC, HHC).

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