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A SEARCH FOR THE PHYSICAL CONTENT OF LUDERS' RULE*

ABSTRACT. An interpretation of quantum mechanics that rejects hidden variables has to say something about the way measurement can be understood as a transformation on states of individual systems, and that leads to the core of the interpretive problems posed by Luders' projection rule: What, if any, is its physical content? In this paper I explore one suggestion which is implicit in usual interpretations of the rule and show that this view does not stand on solid ground. In the process, important aspects of the role played by the projection postulate in the conceptual structure of quantum mechanics will be clarified. It will be shown in particular that serious objections can be raised against the (often implicit) view that identifies the physical relation of compatibility preserved by Luders' rule with the relation of simultaneous measurability.

1. INTRODUCTION

Preparatory measurements are measurements for which we can conclude on the basis of the result of measurement that the system is (immediately after measurement) in an eigenstate of the eigenvalue corresponding to the given result. Preparatory measurements are first of all experimental processes that select an ensemble to which a quantum (statistical) state can be assigned. Statistical interpretations of the state vector take as fundamental this assignment of a quantum state to (sub)ensembles selected by experimental procedures. But if the existence of hidden variables is denied, then it seems that it should be possible to interpret preparatory measurements in terms of individual state transformations. This can be done by thinking of these measurements as idealized measurement transformations satisfying the so-called *first kind condition*: an immediate repetition of the same measurement gives the same result as the initial measurement. This idealization (taken here for the sake of simplicity) will be assumed here to be non-controversial. Furthermore, it will be assumed here that every measurement can be represented as a first kind measurement: thus, a 'measurement' in this investigation is always to be a 'first kind measurement'.

For measurements of maximal magnitudes (i.e., magnitudes with no degenerate eigenvalue) Von Neumann argued (1955) that the final state is given by the (unique) eigenstate of the eigenvalue representing the

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measurement result. This is Von Neumann's projection rule for the assignment of a final state on (maximal) measurement. For first kind measurements the rule follows immediately from the basic quantum mechanical assumption that if a system has a given eigenvalue with certainty (with probability one) then the system is in an eigenstate of this eigenvalue.

Von Neumann suggested that non-maximal measurements be treated as maximal measurements with less than full information about the (nature of the) measurement. A non-maximal measurement for Von Neumann is a maximal measurement in disguise. Von Neumann's rule then, when applied to non-maximal measurements, would make the choice of the final state after measurement dependent on the maximal magnitude implicitly measured. Luders (1951) was apparently the first to suggest that we also need a principle describing *genuine* non-maximal measurements in quantum mechanics: measurements that supposedly take place independently of the measurement of a maximal magnitude.

As I show in the next section, Luders attempted to justify his proposed rule by construing it as a rule describing a change of statistical state on measurement that generalizes the process of state preparation. Indeed, Luders' projection rule receives a strong justification along these lines. But a very important question remains, a question that is the seed of a long debate on the projection postulate: the problem of characterizing the physical process that Luders' rule is supposed to describe as a transformation on individual systems. If we want to characterize Luders' rule as a rule describing a physically distinguishable class of individual state transformations, what are these transformations and how they do come about physically? I will argue that even though some derivations in the literature suggest, or claim to provide the basis for an interpretation of Luders' rule as a description of individual state transformations, such claims cannot be made precise. Ultimately they seem to rely on a confusion between two different problems of justification, the problem of justifying Luders' rule as a statistical principle and that of justifying Luders' rule as a description of individual measurement transformations.

2. Luders' modification of von Neumann's formula

In terms of ensembles the preparatory measurement of magnitude R with eigenvalues $\{r_k\}$ and spectral decomposition $R=\Sigma_k r_k P_k$ is represented by the following ensemble:

(2.1) $$Z' = \sum_k w_k Z_k$$

where Z_k describes the subensemble of those physical systems for which the eigenvalue r_k is obtained.

For maximal magnitudes formula (2.1) would provide a full description of the change of state on a measurement of R. For measurement of non-maximal magnitudes, however, formula (2.1) is ambiguous since for a degenerate eigenvalue there is not a unique eigenvector but infinitely many eigenvectors satisfying (2.1). Von Neumann suggested that in this case the state of the system after measurement should be described as a mixture of the eigenstates in the eigenspace of the measurement result.

Luders raised two objections against Von Neumann's analysis of non-maximal measurements:

(a) The measurement of a highly degenerate magnitude allows only relatively weak assertions about the given ensemble. The corresponding change of state should thus be correspondingly small, while precisely in this case Von Neumann's proposal provides a very complicated ensemble.

(b) One could expect that just as in the formula (2.1) for the calculation of the result of measurement, also the change of state would depend only on Z and r_k (and R). In particular, that the change of state by measurement of R with result r_k is such that if the initial state is pure then the final state is also pure.

In order to see the point of Luders' first objection consider the extreme case of a measurement of the unity (identity) operator. In this case all vectors are eigenstates of the eigenvalue 1. Von Neumann's proposal would lead us to assign as a final state a mixture in which all vectors are assigned the same weight. But in this case it seems obvious that we should expect no change at all in the initial state of the system.

Luders did not explicitly attempt a derivation of his rule. He argued, instead, that if his generalized projection rule (Luders' rule) is assumed, the commutativity of operators representing given observables is equivalent to the compatibility of those observables. The proof of this equivalence will be called here the *compatibility theorem*. Von Neumann had attempted to prove the equivalence between compatibility and commutativity using a rather questionable characterization of compatibility of observables that required an unwarranted assumption (see Section 6). Luders instead provided a mere abstract definition of compatibility in terms of ensembles and selection of subensembles. On the basis of this he was able to prove the compatibility theorem within the purely statistical (Hilbert space) framework of the theory. Previous to a discussion of the significance of Luders' theorem I start with an outline of his analysis of compatibility.

(2.2) DEFINITION (Luders'). In an ensemble a magnitude R is measured and the subensemble corresponding to a particular measurement value r is selected. Immediately after (so that one can abstract from the change due to dynamical evolution) one measures on this subensemble the magnitude S and the subensemble corresponding to the measured value is selected. The measurement of R and S are called *mutually compatible* when a subsequent measurement of R [on the second ensemble] gives the result r with certainty.

(2.3) DEFINITION (Luders'). Two measurements of R and S are *mutually compatible* when the interposition of the measurement of R [in between two measurements of S] without selection of a subensemble does not alter the result of the S measurement.

The above definitions characterize the concepts of compatibility for magnitudes with and without selection of subensembles (Luders 1951, p. 323). *Two magnitudes R and S are mutually compatible (or-noninterfering) if there are mutually compatible measurements of R and S.* Notice that the above definitions of compatibility apply in the first place to (preparatory) measurements characterized statistically. But whereas the first definition is framed in a purely statistical language of ensembles and selection of subensembles, the second definition is more neutral in its formulation lending itself (contrary to the first), at least in principle, to use as a definition of compatibility in an individual state interpretation. I will refer to Luders' definition of the compatibility relation as L-compatibility. When a precise reference to one of the two definitions of compatibility is needed I will distinguish between L_1-compatibility and L_2-compatibility. Since I am interested in individual interpretations of Luders' rule I will usually focus on L_2-compatibility.

Guided by the two objections presented above, Luders argued that Von Neumann's rule should be replaced by the following formulae:

(1) If an ensemble initially in state Z, is measured for magnitude R with result r_k, then after the measurement the state of the corresponding subensemble represented by P_k is

(2.4) $$Z' = P_k Z P_k .$$

(2) After the same measurement, the state of the whole (original) ensemble is

(2.5) $$Z' = \sum_k P_k Z P_k .$$

It is easy to see that Z' fulfills the desired mathematical requirements (Z' is Hermitian and positive). Luders' main theorem follows.

(2.6) THEOREM (Luders). If after measurement (without selection) of an ensemble in state Z the final state is given by Luders' rule (formula (2.5)) then the observables R and S are (L_2-) compatible if and only if the operators representing R and S commute.

The proof of this theorem can be found in Luders (1951).

Luders' theorem is the core of Luders' justification of his proposed formula. Luders' theorem shows that if one assumes that his formula describes the state after measurement, the physical significance of the relation of compatibility can be made precise in terms of the existence of non-interfering measurements (interpreted statistically). If the problem is that of providing a justification for the selection of Luders' rule among all possible rules that select a pure state after measurement of a pure state, Luders' theorem provides a strong justification. It is not difficult to see that if we take any other projection rule instead of Luders' projection rule, L-compatibility is not equivalent to commutativity. The possibility exists that a contrived modification of compatibility would still allow a proof of Luders' theorem to go through. But even this far fetched possibility is taken care of by a derivation of Luders' rule to be reviewed later. *The problem of justification referred to above, the problem to which Luders' theorem provides a convincing answer, is the problem of finding the rule that selects the 'right' statistics on remeasurement of a magnitude (after a non-maximal measurement).*

An additional and more elaborated justification will be obtained from the different derivations to be reviewed later. Luders' rule will be shown to follow from some basic principle of 'minimal change' or 'non-interference'. *But this problem must be distinguished from the problem of interpretation of Luders' rule as a description of the process of individual measurement.*

It is possible that there is no underlying physical process to be interpreted as an instance of an individual measurement. This would be a very bad state of affairs for anyone believing in an individual state interpretation. At the very least one should be able to give a plausible account of Luders' rule as a description of individual processes. Elsewhere (1987) I argue that once an important presupposition of individual state interpretations is dropped (that an individual state is represented by the set of properties to which a given state vector assigns probability one), it is possible to provide a clear and precise interpretation of Luders' rule. But the first step is the separation of different problems.

One can distinguish two types of derivations. On the one hand we have derivations that assume the Hilbert space structure as a whole and only add to this whatever assumption is considered needed for a derivation of Luders' rule with a desired semantical interpretation. On the other hand one could think of derivations that do not include the Hilbert space structure as a whole among its premises, but rather try to provide a derivation of Luders' rule from individually motivated principles.

The rationale of such distinction is the following. Since the Hilbert space structure provides a non-controversial formulation of the statistical structure of the theory, it turns out to be an easy task to provide a purely statistical interpretation of Luders' rule based on derivations in which the Hilbert space structure is assumed from the onset. Such statistical interpretations do not require the presupposition that there is a process of individual measurement described by Luders' rule. Therefore, in order to justify an individual state interpretation one cannot assume indiscriminately the Hilbert space structure as a whole. The justification of individual state interpretations of Luders' rule seems to require the use of principles that can be justified as fundamental principles describing the behavior of individual systems. That is, an individual state interpretation of Luders' rule requires the derivation of Luders' rule from principles which are semantically relevant for states and processes of individual systems.

It must be emphasized that the above distinction between statistical and individual state interpretations concerns the physical interpretation of Luders' rule. The issue is the justification of whatever physical content we attribute to the rule via the justification of the semantic interpretation of the principles from which it is derived. Since the above distinctions are not usually made, claims about the different problems of justification are usually conflated.

3. DERIVATIONS OF LUDERS' RULE

The question whether Luders' rule can be derived in quantum mechanics is a long standing issue and it is often taken as the focus of discussion on the projection postulate. Often outlines or suggestions for a putative derivation are given, but the details (and often not only the details) of the derivations are missing. More often Luders' rule is only motivated as a 'convenient idealization'. Jauch (1968) for example motivates Luders' rule by noticing that for projections P with one-dimensional range and state (represented by operator) Z, $PZP = (TrZP)P$. After a preparatory measurement then, the state describing the system is given by $Z' = \Sigma_k P_k Z P_k$ for any one-dimensional projections P_k, where $R = \Sigma_k r_k P_k$ is the spectral decomposition of a magnitude R with eigenvalues r_k. Now Jauch (1968) says:

> It is therefore convenient to introduce the notion of the ideal measurement which affects the state in a minimal way, and for which the state after measurement is still given by the formula [$Z' = \Sigma_k P_k Z P_k$] but without the requirement that the projection operators be one-dimensional. (p. 166)

A rather unconvincing motivation for such a controversial principle. Jauch's observation, however, seems to suggest a derivation once we notice the formal analogy made explicit by Jauch between the description offered by Luders' rule and the description of preparatory measurements. Such analogy is cashed in for example in Herbut's derivation to be reviewed later (Section 3.7).

Attending to the descriptive account of measurement implicit in the derivations one can distinguish two different sorts of derivations. One sort is framed in the language of statistical operators and selection of subensembles. Another sort of derivation uses the language of eigenvalues or 'sharp-values' instead of the language of selection of subensembles in the description of measurement. There is of course not a clear cut distinction between what we can call 'statistical' and 'sharp value' derivations. A derivation can use a mixed language of statistical operators and 'sharp values'. But as we shall see it is worth drawing the distinction as a preliminary classification of the available derivations. Sharp value derivations suggest the sort of interpretation of Luders' rule which if justified could (at least in principle) provide us with an individual-state derivation of Luders' rule. Next I lay out several derivations of Luders' rule and make explicit the presuppositions involved.

Luders seems to have shared Von Neumann's idea that what he was proposing was a basic principle on measurement, required as an indispensable postulate for the interpretation of the theory. He seemed to be satisfied with a clear formulation of his principle within the statistical framework of the theory and the justification of his proposal on the basis of its consequences for the interpretation of the mathematical structure of Hilbert spaces. But it is not difficult to see how Luders' formal results presented in (1951) can be used (as the following theorem shows) to generate a derivation of the projection rule, in the Hilbert space framework under a very plausible assumption of compatibility.

(3.4) THEOREM (Luders' converse). For preparatory measurements, if it is assumed that whenever two operators commute the corresponding magnitudes are L-compatible then the change of state by a measurement is described by Luders' rule.

The proof of the theorem uses the following simple result from operator theory:

(3.5) LEMMA. Let P and Q be two commuting operators. Suppose $P = \sum_n P_n$, and $Q = \sum_n P_n Q P_n$.

The proof of Luders' converse follows.

Suppose that P and Q are two commuting magnitudes and that the initial state is Z, then after measurement of P $TrZ'Q = TrZQ$ since P and Q are L-compatible. Now,

$$TrZQ = TrZ\left(\sum_n P_nQP_n\right) \qquad \text{By lemma 3.5}$$

$$= TrQ\left(\sum_n P_nZP_n\right) \qquad \text{by properties of trace}$$

thus,

$$TrZ'Q = Tr\left(\sum_n P_nZP_n\right)Q$$

since Q is arbitrary (see Von Neumann 1955, p.188)

$$Z' = \sum_n P_nZP_n$$

Q.E.D.

This last proof is implicit in Furry (1965).

Herbut (1969) argued similarly to Luders that if we take the measurement of a non-maximal magnitude R in the sense of Von Neumann, as the implicit measurement of a maximal magnitude M we will in general overmeasure R in the sense that measuring M gives in general finer subensembles than what one would expect on the ground of measuring R alone. Herbut claimed that what is required is to develop the idea of the 'minimal measurement of an observable'. For any operators R and S belonging to the operator Hilbert space there is a unique concept of distance defined by

(3.6) $\quad d(R, S) = \| R - S \| = (R - S, R - S)^{1/2}$.

Herbut used this metric to put the idea of minimal (preparatory) measurement of an observable R in precise mathematical form: the minimal measurement of an observable R should transform an arbitrary quantum state Z into a state Z' which is as close as possible to Z in this metric. This is the content of Herbut's first theorem.

(3.7) HERBUT'S FIRST THEOREM. For preparatory measurements, if the change of state by measurement is minimal in the metric of the operator Hilbert space f (as defined above, p. 5) then Z' is given by Luders' rule, that is $Z' = \sum_n P_nZP_n$.

For the proof see Herbut (1969).

In 1974, Herbut provided a different derivation of Luders' rule. According to Herbut one can characterize Luders' rule as describing minimally disturbing measurements via the concept (Dirac's) of 'complete observable'. Given a non-maximal observable R one can form a sequence of compatible observables $R, S, \ldots$ to achieve a complete set. The measurement of this sequence of observables generates a corresponding sequence of eigenvalues (sharp values) $r, s, \ldots$ which enables one to arrive at a pure state: for this accumulation of sharp values it is indispensable that "each measurement should preserve the ones already acquired by the system in the previous measurement of compatible observables" (Herbut 1974, p. 196). Thus, he claimed, we are led to impose the following requirement on preparatory measurements:

> *Requirement H*: If a physical system in state Z has the sharp value d of an observable D and if this observable is compatible with the measured observable R, then whichever result r of the measured observable R is obtained, the N systems having produced this result have to be in a state Z in which the sharp value d of D is preserved.

With this requirement Herbut proved the following theorem.

(3.8) HERBUT'S SECOND THEOREM. (A) If the preparatory measurement of an observable R (with spectral decomposition $R = \sum_n r_n P_n$) satisfies requirement H then an arbitrary state Z is changed into the state Z' given by Luders' rule. (B) If $TrZP_n \geq 0$ implies Luders' rule for the change of state by measurement then requirement H is valid.

Notice that requirement H could be more precisely formulated by assuming the commutativity (not the compatibility) of the corresponding observables. This makes clear in particular that Herbut's Theorem (B) is a sharp-value reformulation of Luders' rule. As Herbut points out, his derivation of Luders' rule is an improvement over Luders' own formulation because the class of observables having a sharp value d in the initial state and being compatible with R is a very restricted subclass of the class of all observables compatible with R. But even more important for us is the fact that *Herbut's theorem provides a way of extending Luders' characterization of compatibility (and his formula for the change of state by measurement) to (states of) individual systems*. Requirement H can be seen as a reformulation of the relation of compatibility that can be applied to individual systems since sharp values can be interpreted as belonging to each system in the ensemble (corresponding to the given value). Implicit in Herbut's approach is then the following characterization of compatibility for magnitudes of individual systems: two magnitudes are compatible if sharp values are preserved by measurement. Instead of going into the proof of Herbut's theorem I will present a simplified derivation of Luders' rule given by Stairs in 1982. Whereas Herbut formulated his result in a mixed language of statistical states and sharp values Stairs provides a derivation using the definition of compatibility implicit in Herbut's account in a formulation of quantum mechanics entirely within the framework of individual system interpretations. Stairs' derivation can be seen as a paradigm of what I call 'sharp-value' derivations.

Stairs' derivation of Luders' rule. Stairs characterizes the compatibility of magnitudes in terms of non-interfering measurements. Two magnitudes are compatible if there exist non-interfering measurements of the magnitudes. What this means in idealized experimental terms is that if Q_1 and Q_2 are two compatible magnitudes then there exists measurements of Q_1 and Q_2 such that

(i) if performed simultaneously the system will be left in a common $Q_1 - Q_2$ eigenstate, and
(ii) if performed in sequence $(Q_1, Q_2, Q_1, \ldots)$ the measured values will be stable. Stairs then characterizes ideal measurements as those which are non-interfering for compatible magnitudes. Stairs proves that if one accepts the possibility of sequences of measurements on a single system and agrees that ideal measurements of compatible magnitudes should be non-interfering one can derive Luders' rule.

Following Stairs we proceed to the derivation of Luders' rule in terms of an example. Here I follow Teller's 'no frills' reformulation in (1983). Suppose our system is represented by a 3-dimensional Hilbert space H. A state of the system is then a linear combination of an arbitrary basis for H, say $\{\alpha_1, \alpha_2, \alpha_3\}$. Consider the magnitudes (represented by) $A = a_1 P_{\alpha_1} + a_2 P_{\alpha_2} + a_3 P_{\alpha_3}$ $B = b_1 P_{\alpha_1} + b_2(P_{\alpha_2} + P_{\alpha_3})$. If the initial state of the system is ϕ and we measure B getting result b_2, the resultant state must be an eigenvector in the $\alpha_2 - \alpha_3$ plane. Now consider a magnitude D with two eigenvalues d_1 and d_2 such that the eigenspace corresponding to d_1 is the $P_{b_2} - \alpha_1$ plane and the eigenspace corresponding to d_2 is the orthogonal plane. Measure magnitude D, compatible with B, and suppose the result is d_1. The state after measurement lies in the $\alpha_2 - \alpha_3$ plane. But on the assumption that compatible (sharp) values are respected it must also lie in the $P^{\phi}_{b_2} - \alpha_1$ plane. Thus, the final state must be in the intersection of both planes and this is just P_{b_2}.

A quantum logical derivation. In the quantum logical framework Luders' rule can be derived via the equivalence relation generated by the Sasaki-hook. Suppose that initial state is a and we measure non-maximal magnitude M with result (represented by proposition) r. In general there will be an infinite number of possible state transformations describing this measurement. The state transformation described by Luders' rule is singled out, however, by the fact that, given that a is the initial state then r and the state selected by Luders' rule, $L(a, r)$, are equivalent under the following equivalence relation:

$$\text{(E)} \qquad x \sim y \qquad \text{if and only if} \qquad (x \supset y) \wedge (y \supset x) = 1$$

where $\supset$ stands for the Sasaki hook: $x \supset y = x^{\perp} \vee (x \wedge y)$. For details and discussion of this derivation see Friedman and Putnam (1978), Hellman (1981) and Stairs (1982). Friedman and Putnam argued that the equivalence relation (E) is a logical equivalence relation already built into quantum logic (generated by the quantum logical biconditional) which allows the derivation of the projection postulate (in 'real quantum logic') without any additional postulates.

4. THE PRESUPPOSITIONS INVOLVED IN THE DERIVATIONS

At first sight it is not quite clear what assumptions on the compatibility relation are required by Herbut's second theorem and Stairs derivation. One could be inclined to think, from the way in which they describe the relation of compatibility, that the fact that two magnitudes are compatible *implies* the existence of the sharp values that are to be preserved in the process of measurement. But to assume this strong characterization of compatibility at this point would be inappropriate. Remember that Luders' theorem only shows the equivalence between

L-compatibility (compatibility as defined by Luders) and the commutativity of the corresponding operators under the assumption of Luders' formula for the change of state.

It is not difficult to check, however, that Herbut's requirement H only requires that commutativity is a sufficient condition for compatibility. In order to see this simply replace in requirement H the assumption that R and D are compatible by the assertion that R and D commute, and assume that commutativity is a sufficient condition for compatibility. Similarly, notice that the first condition that Stairs put forward as characterizing compatibility (condition (i)) is not really needed in Stairs' derivation, all that is needed, again, is that commutativity is a sufficient condition for simultaneous measurability (which implicitly is identified with physical compatibility). Stairs' derivation is thus a version of Luders' converse in which (as in Herbut's second derivation) compatibility is characterized in terms of preservation of sharp values.

Luders' converse theorem, as well as Stairs' and Herbut's second theorem only need to assume that commutativity is a sufficient condition for compatibility. As an assumption on a physical relation of compatibility this should be uncontroversial. It would make little sense to think of the possibility of commuting magnitudes that are not compatible. But the physical interpretation of Luders' rule as a state transformation respecting sharp values also implicitly assumes more than what is granted by the derivations.

Notice that in Luders' converse theorem when we talk of the change of state by measurement we mean a measurement that is 'non-interfering'. If we think of Luders' rule as an algorithm generating the statistical state after measurement, 'non-interference' has a straightforward interpretation in terms of preservation of relative probabilities or sharp-values (see discussion below). But when Luders' converse theorem is interpreted as referring to individual processes it is not even clear, to start with, what is meant by a 'non-interfering' measurement. If we assume that the compatibility of magnitudes implies the commutativity of the corresponding observables we would have, at least in principle, a clear notion of non-interference. For in this case 'respecting compatibility', could mean nothing more and nothing less than what is stated in Herbut's requirement, that sharp values are preserved (as Von Neumann has shown in (1955) commuting magnitudes have all of their eigenvalues in common). *But if the physical relation of compatibility is stronger than the commutativity of the corresponding observables the* *idea of 'respecting compatibility' would need to be reformulated since now preservation of sharp values could not be used to fully describe preservation of a physical relation of compatibility stronger than T-compatibility.* It is possible to think of weakening the idea of sharp value so that sharp values can arise also in connection with non-commuting magnitudes, but it is far from clear how this idea could be implemented in the framework of Luders' type derivations.

One could think of using Herbut's second derivation to improve on this state of affairs. Indeed, as we have seen, Herbut's derivation is an improvement insofar as it requires less information to be preserved in measurement. But when Luders' rule is seen as a description of what happens at the level of individual processes very similar questions about the nature of non-interference arise anew. Herbut's second derivation allows for an alternative reading of non-interference as a description of a sequence of observables and a corresponding sequence of measurements zeroing on a pure state. This approach provides a way of describing after the fact a plausible way of arriving at the state selected by Luders' rule through a series of measurements satisfying some adequacy condition. But that sort of analysis cannot provide an answer to the question of why a single measurement ends up in the state described by Luders' rule. Luders' rule would describe only the limit of an (infinite) series of physical transformations, not the result of a single transformation. Nor would this analysis explain why Luders' rule comes up with the 'right' final state for remeasurement.

Furthermore, sharp value derivations have to confront an additional problem. Let us assume for the sake of the argument that for each pair of L-compatible magnitudes there is an individual state transformation corresponding to the non-interfering measurement of one of the magnitudes, as would be required by Definition (2.3). In order to accept the interpretation suggested by sharp-value derivations as a description of individual state transformations, we would need to assume that there is a transformation that is non-interfering 'simultaneously' with respect to all relevant compatible pairs of magnitudes. This is an assumption that is not obviously justified nor does it seem reasonable by itself. On what grounds do we expect this transformation to exist? Why would we expect that an 'ideal' (non-interfering) transformation exists at all in the sense required by these derivations interpreted in terms of individual systems? According to Luders' type derivations there is no underlying measurement of an all embracing maximal magnitude that would justify such a belief. This question would only be more pressing if physical compatibility is a stronger relation than (the one represented by) the commutativity of the corresponding magnitudes.

The 'real' quantum logical derivation of Friedman and Putnam claims that the concept of ideal measurement is a purely logical concept. As Hellman in (1981) has pointed out, however, it is not clear how to represent conditional reasoning in quantum logic and the Sasaki-hook provides only one of many possibilities for doing so. The equivalence (E) presupposes the postulation of the Sasaki hook as generating a privileged relation in quantum logic. This postulate, however, cannot be justified but by appealing to principles that lie beyond the 'logic' of the orthomodular structure. Here it is enough to notice that this derivation presupposes the Hilbert space metric and to that extent the derivation is redundant, since Luders' rule can clearly be derived under this assumption without additional interpretational baggage (as Herbut's first derivation shows).

Friedman and Putnam's point is that a quantum logical interpretation allows us to derive Luders' rule whereas the Copenhagen interpretation has to assume it as an ad hoc principle. But the problem of justifying Luders' rule that they claim to address is the problem of trying to characterize individual processes responsible for quantum statistics in such a way that the statistics can be grounded, at least in principle, on such individual processes. *To the extent that quantum logic considers the statistical structure as directly reflecting the structure of individual systems (events) Luders' rule is simply assumed in the process.* That is because this statistical structure includes the Hilbert space metric and minimal disturbance in that metric is equivalent to the choice of the Sasaki conditional (see Hardegree 1976). *Quantum logic cannot claim any advantage from the possibility of deriving Luders' rule in quantum logic. This derivation is just a lattice theoretical formulation of a mathematical theorem (Herbut's first theorem) which is available to all interpretations.*

If all we are concerned with is the statistics of measurement results, Herbut's derivation together with its statistical interpretation provides a solid justification of Luders' rule. This is an important issue and one that is definitely solved. The problem that is left is how to interpret Luders' rule, if possible, as a description of a process corresponding to a change in the state of an individual system, and how to justify such an interpretation.

Luders' converse theorem suggests a way. Herbut's second theorem and Stairs' derivation can be seen as attempts to elaborate on this suggestion by providing derivations in terms of the accumulation or preservation of 'sharp values'. But I am claiming that these theorems (as individual state derivations) fail in that the interpretation of Luders' rule they try to justify is suggested by the statistical interpretation, but there do not seem to be a firm basis for such an interpretation at the level of individual processes. In particular the usual interpretation assumes something that is not granted by the derivation, namely, that the relation of compatibility respected by Luders' rule is (what I call in the next section) T-compatibility. As it will be shown in the second part of this paper such an assumption does not seem to be justifiable. A natural construal of a relation of simultaneous measurability in quantum mechanics leads us to a relation of physical compatibility that is not the relation of compatibility presupposed by usual interpretations.

We have seen that different derivations of Luders' rule require different presuppositions. Derivations most easily construed as assuming statistical interpretations of the quantum mechanical state (the paradigm of which is Herbut's first theorem) are rather simple and show that the interpretation of the metric of operators provides a statistical interpretation of Luders' rule. Sharp value derivations on the other hand suggest (and usually are taken to support) the stronger claim that they provide a physical derivation (interpretation) of Luders' rule that takes a single system interpretation of state. I have argued that these derivations do not justify such a strong physical interpretation of Luders' rule. In the following section I provide a formal analysis of the relation of compatibility and the concept of ideality as they are implicit in the derivations reviewed above. This will provide us with a 'map' to guide us through the maze of different versions of Luders' rule, allowing us to draw general conclusions and making more precise our earlier discussion. This analysis as well as its relevance for my discussion of Luders' rule was suggested to me by G. Hellman.

5. A FORMAL ANALYSIS OF COMPATIBILITY

Luders' rule is supposed to 'respect' or 'preserve' the relation of compatibility. If this is taken as statistically characterizing the physical content of Luders' rule (as Luders seems to have intended) there should be no additional quarrel. The derivations we have reviewed above provide a firm basis for such a view. But the usual view is that Luders' rule describes a privileged class of state transformations that play a fundamental role in the interpretation of the theory and for justifying this view it is not sufficient to take Luders' rule to be a rule describing state transformations respecting compatibility. The relation of compatibility at stake must be physically interpreted in such a way that the preservation of this relation by Luders' rule can be justified.

There are several alternative ways of defining a relation of compatibility satisfying some minimal requirements. Some of these relations are (or might be) physically relevant in the Hilbert space formulation of quantum mechanics. In (1978) Hardegree has argued for the importance of a relation of 'partial compatibility' in order to understand certain aspects of the process of measurement. This last relation corresponds to what I call *P*-compatibility (see Definition (5.1) below). Below M_i are taken to be magnitudes, that is, orthonormal sets of eigenvectors. The closed subspaces generated by a magnitude M_i will also be denoted by M_i.

(5.1) DEFINITION. $M_1 \underset{P}{\leftrightarrow} M_2 \equiv_{\text{def}} M_1$ is *P*-compatible with M_2 if M_1 and M_2 have some eigenvectors in common.

(5.2) DEFINITION. $M_1 \underset{T}{\leftrightarrow} M \equiv_{\text{def}} M_1$ is *T*-compatible with M_2 if M_1 and M_2 have all eigenvectors in common.

(5.3) THEOREM

(i) $$M_1 \underset{T}{\leftrightarrow} M_2 \Rightarrow M_1 \underset{P}{\leftrightarrow} M_2.$$

(ii) $$M_1 \underset{P}{\leftrightarrow} M_2 \not\Rightarrow M_1 \underset{T}{\leftrightarrow} M_2.$$

A measurement transformation is a transformation $T(a_i, r) = a_f$ where a_i is (an eigenvector representing) the initial state, r is the result of measuring a magnitude M and a_f is an eigenstate of the measurement result. We say that a *measurement preserves (respects) a magnitude X* if for $x \in X$, $x \in [a_i)$ then $x \in [a_f)$. Here $[x)$ is the principal filter generated by the element (ray) x (i.e., $[x)$ is the set $\{y;\ x \subseteq y$, where the relation $\subseteq$ stands for the relation of inclusion among closed subspaces$\}$. This definition makes use of the usual ambiguity between elements of a Hilbert space and the generated one-dimensional subspaces (rays).

(5.4) DEFINITION. A measurement of M, on a system in state a_i is P-ideal if it preserves all X, such that X is a magnitude P-compatible with M.

(5.5) DEFINITION. A measurement of M, on a system in state a_i is T-ideal if it preserves all X, such that X is a magnitude T-compatible with M.

P-ideality and T-ideality are special cases of a more general (mathematically and physically feasible) definition of ideality.

(5.6) DEFINITION. A W-ideal measurement is a measurement that preserves all X, such that X is a T-compatible magnitude with M and (that also preserves) a given set of magnitudes $\{X_i\} = W$ such that $X_i \underset{P}{\rightarrow} M$.

A W-ideal measurement is a T-ideal measurement which in addition preserves *some* partially compatible magnitudes. Even though the idea of W-ideal measurement (and the implicit relation of W-compatibility) might appear strange at first sight, consideration of this version of ideality cannot be dismissed on a priori grounds. Physical reasons might be invoked to select the set $\{X_i\}$. One could very well make sense of this idea in analogy with the existence of superselection rules; since as in the case of the existence of superselection rules one could argue that there is a set of magnitudes which have a privileged status in that they do not only require T-preservation by measurement but also P-preservation. Suppose M is such a quantity. The idea would be that if we measure magnitude M then an ideal measurement preserves not only all T-compatible magnitudes with M but it also preserves M' if M' is P-compatible with M. We assume here, however, that, in accordance with our discussion in Section 4, any (physically acceptable) concept of ideal measurement includes T-ideality as special case.

The paradigmatic candidate for a concept of a physically ideal measurement is suggested by Von Neumann's theorem (see 1955, chap. III): Two magnitudes are simultaneously measurable if and only if their corresponding operators commute.

(5.7) DEFINITION. A measurement of M with initial state a is S-ideal if it preserves all X, such that X is a magnitude simultaneously measurable with M.

(5.8) THEOREM. For any R-concept of ideal measurement, where R can stand for P, T, W, or S, M is R-ideal$\Rightarrow M$ is T-ideal.

The proof follows immediately from definitions above and from the fact that whenever A and B are T-compatible then A and B are simultaneously measurable. Notice that this theorem only requires the 'easy' direction of Von Neumann's theorem.

(5.9) DEFINITION SCHEMA. R-Luders $\equiv_{\text{def}} \forall S$, S is a measurement state transformation, S is R-ideal$\rightarrow S$ obeys Luders' rule.

The following is then immediate:

(5.10) THEOREM. T-Luders$\rightarrow R$-Luders for $R = P$, W, or S.

This theorem provides a more precise statement in terms of ideal measurements of the core of the sharp value derivations as these have been formulated above.

Now I can make some of my earlier criticisms of the usual interpretation more precise. Luders' rule describes ideal measurements independently of the physical interpretation of the concept of ideal measurement as long as an ideal measurement satisfies the minimal requirement that T-compatible magnitudes are preserved. But, if not on the basis of Luders' rule, on what basis do we select a physically privileged relation of compatibility which is supposed to be preserved by individual measurement transformations? More importantly, how can we justify such a selection of a compatibility relation without merely postulating in an ad hoc manner the existence of such relation and the fact that ideal (first-kind) measurements respect it, as opposed to any other possible relation? As we have seen, the usual arguments for the identification of such a relation in terms of minimal disturbance of 'sharp values' are far from conclusive. Furthermore, the usual interpretation seems to be committed to the unwarranted assumption that the relation to be respected is T-compatibility. Because if the relation to be respected is stronger than T-compatibility the idea of characterizing Luders' rule in terms of preservation of sharp values loses its intuitive appeal. Next I explore the possibility of emending the usual view by arguing that indeed the physical relation of T-compatibility can be singled out as the fundamental physical relation of compatibility that Luders' rule is supposed to respect, by identifying T-compatibility with *the* physical relation of simultaneous measurement in quantum mechanics.

6. SIMULTANEOUS MEASURABILITY AND LUDERS' RULE

If we are able to interpret T-compatibility as a fundamental physical relation *for individual systems* then we would in principle have a way of understanding the physical significance of Luders' rule. Von Neumann's theorem of simultaneous measurability (in 1955, chap. III) provides what is usually taken as the basis for the view that T-compatibility is to be identified with simultaneous measurability. In this theorem Von Neumann proved the equivalence between an implicit notion of simultaneous measurability for observables and the commutativity of the corresponding observables. If Von Neumann's proof could be accepted in reference to individual states then it would seem we would have a way, at least in principle, of understanding the physical content of Luders' rule in terms of individual state transformations that respect the physically fundamental relation of simultaneous measurability. We shall see however that Von Neumann's proof of his theorem relies on a highly questionable assumption which can only be justified by assuming from the outset the Hilbert space structure. An alternative proof of Von Neumann's theorem by T. Jordan appears to bypass the criticism raised against Von Neumann's proof, but Jordan's proof will be shown to be arbitrarily restrictive. Removing this arbitrariness leads to a theory of joint measurements for non-compatible observables in a sense which directly undermines the usual characterization of Luders' rule.

For the proof of his compatibility theorem Von Neumann argued that, given physical magnitudes (quantities) A and B, a new physical magnitude $A + B$ can be constructed on the basis of the additivity of the eigenvalues of the corresponding observables. The new physical magnitude $A + B$ corresponds to an operator with eigenvalues $a + b$. This assumes what can be called 'the postulate of additivity'. The postulate of additivity is the claim that the addition of the results of measurement provides ground for the definition of a compound magnitude $A + B$ which is supposed to represent the physical process of simultaneous measurement. Once this postulate is accepted the physical problem of characterizing simultaneous measurability is readily transformed into a problem in Hilbert space theory, and the proof of the theorem follows as a rather simple derivation from Von Neumann's axioms.

Bell has argued, in the context of a discussion of hidden variable theories (but see also Bell 1982), that if two operators S_x, S_y (with eigenvalues s_x and s_y) do not commute then the sum observable represented by the operator $S_x + S_y$ need not have eigenvalues $s_x + s_y$. Bell presented the following example in (1982): For spin-1/2 particles: Let P and Q be components of spin angular momentum in perpendicular directions, $P = S_x$, $Q = S_y$ and let O be the component along an intermediate direction, $O = (P + Q)/\sqrt{2}$. The eigenvalues of O, P, Q are all magnitude 1/2 whereas Von Neumann's postulate would require $(\pm 1/2 \pm 1/2)\sqrt{2} = \pm 1/\sqrt{2}$. Thus, we have to conclude that Von Neumann begs the question when he assumes his additivity postulate, since only operators represented by commuting operators must satisfy his axioms – others need not. Thus the interpretation of the physical relation of compatibility as this would be required by the usual interpretation of Luders' rule is not settled by Von Neumann.

T. Jordan takes in (1969) a different approach to the concept of simultaneous measurability. On the basis of a simple definition of simultaneous measurability this approach seems to bypass the problems with Von Neumann's theorem. *Jordan provides the following explicit necessary condition for simultaneous (joint) measurability*: suppose A and B represent quantities that are simultaneously measurable with unlimited precision. Then for any real numbers x and y measurements can determine if the values of the quantities represented by A and B are either

$$\begin{array}{lll} \leq x & \text{and} & \leq y, \\ \leq x & \text{and} & > y, \\ > x & \text{and} & \leq y, \\ > x & \text{and} & > y. \end{array}$$

Jordan constructs a real measurable quantity with values 1, 2, 3, 4 corresponding to these four mutually exclusive possibilities. This quantity is represented by the operator

$$I_1 + 2I_2 + 3I_3 + 4I_4$$

where I_1, I_2, I_3, I_4 are mutually orthogonal projection operators such that

$$I_1 + I_2 + I_3 + I_4 = 1.$$

Now he proceeds to prove the following theorem.

(6.1) THEOREM. (Real) quantities that are simultaneously measurable with unlimited precision are represented by Hermitian operators A,B, such that $[A,B]=0$.

The proof can be found in Jordan (1969, p. 87). It is a straightforward proof in the Hilbert space framework. Jordan seems to provide a proof of Von Neumann's theorem which would avoid the difficulties we found with Van Neumann's original proof. But that is not the end of the story.[1]

Following the analysis presented by Muynck et al., in (1979), I will show that Jordan's assumption that magnitudes can be measured with unlimited precision is arbitrarily restrictive and that removing it leads to the possibility of formulating joint measurement schemes for non-compatible measurements in a sense that undermines the identification of T-compatibility with simultaneous measurability in the strong sense required by usual interpretations of Luders' rule.

Let us suppose that a system S described by a state function interacts with two measuring apparatuses (for observables) A and B. Let us symbolize the measurement procedure by $T_{A,B}$. The magnitudes are represented by Hermitian operators A and B with eigenvalues and eigenvectors given by

$$A_{\alpha_m} = a_m \alpha_m$$
$$B_{\beta_n} = b_n \beta_n.$$

Following the usual (Von Neumann's) analysis of the evolution of the combined system on measurement, the evolution of system S plus the A- and B-apparatuses is assumed to take place according to the scheme

(M1) $$\psi_i = \psi \otimes x_0 \otimes \xi_0 \xrightarrow{T_{A,B}} \psi_f = \sum_{l,m,n} s_{lmn}(\psi, T_{A,B}) \phi_l \otimes \theta_m \otimes n_n$$

in which $\{\phi_l\}$, $\{\theta_m\}$, $\{n_n\}$ are complete orthonormal sets (in Hilbert space

H) representing S, A and B respectively. χ_0 and ξ_0 are the initial states of the A and B meters respectively. $S_{lmn}(\psi, T_{A,B})$ are coefficients depending on the initial state ψ_i of the object system and possibly on the measurement procedure $T_{A,B}$.

The joint probability distribution $W(a_m, b_n; \psi; T_{A,B})$ for A and B may be then defined as the expectation value of the projection operators $P_m \otimes Q_n \equiv |\theta_m\rangle\langle Q_m|\eta_n\rangle\langle\eta_n\rangle$ in the final state ψ_f, thus yielding

(M2) $$W(a_m, b_n; \psi; T_{A,B}) = \sum |s_{lmn}(\psi, T_{A,B})|^2$$

$$\sum_{m,n} W(a_m, b_n; \psi; T_{A,B}) = 1.$$

In order to obtain a physically satisfactory definition of joint measurement Muynck et al., want to impose restrictions on the scheme $T_{A,B}$. One obvious restriction is that the transformation $T_{A,B}$, being a quantum mechanical process, should be linear. What additional restrictions can be imposed on the scheme? Muynck et al., investigate the following restriction: in the final state the relevant measuring apparatus should have the corresponding pointer position with certainty if S is initially in a state described by an eigenfunction of A and B. This represents the requirement that the A-measurement is not disturbed by the B-measurement (and vice versa). Omitting in the notation the dependence on $T_{A,B}$ this requirement is expressed in the coefficients by the relations

(M3) $$s_{lmn}(\alpha_r) = \delta_{mr}(s_{lrn}(\alpha_r)); \; s_{lmn}(\beta_s) = \delta_{ns}(s_{lms}(\beta_s))$$

or equivalently

(M3′) $$s_{lmn}(\psi) = \langle\alpha_m|\psi\rangle s_{lmn}(\alpha_m) = \langle\beta_n|\psi_{slmn}(\beta_n).$$

Park and Margenau (in 1973) have shown that a scheme obeying this requirement is inconsistent if A and B do not commute. They draw the conclusion that a scheme (M1) for joint measurement is not suitable for expressing joint measurement of observables with non-commuting operators. As Muynck et al., show, however, it is not scheme (M1) that is to blame but rather the requirement (M3).

From (M2) and (M3) it follows that

(M4a) $$W(a_m, b_n, \psi) = |\langle\alpha_m, \psi\rangle|^2 W(a_m, b_n; \alpha_m)$$

(M4b) $$= |\langle\beta_n|\psi\rangle|^2 W(a_m, b_n; \beta_n)$$

which leads to the marginal distributions

(M5)

$$\sum_n W(a_m, b_n; \psi) = |\langle\alpha_m|\psi\rangle|^2$$

$$\sum_m W(a_m, b_n; \psi) = |\langle\beta_n|\psi\rangle|^2$$

and thus, from (M4) and (M5) we get

(M6) $$W(a_m, b_n, \alpha_m) = W(a_m, b_n; \beta_n) = |\langle\alpha_m|\beta_n\rangle|^2.$$

The joint probability distribution $W(a_m, b_n; \psi)$ is then fully determined by (M4) and (M6) *provided* (M4a) and (M4b) are equal. As Bub has shown (see Bub 1974, chap. 4) this requirement can only be fulfilled if $[A, B] = 0$, thus accounting for the alleged inconsistency of the scheme (M1). It is then clear that if joint probability distributions are to be defined in quantum mechanics for non-commuting magnitudes either (M1) or (M4) must be weakened. It is not difficult to see that (M1) by itself does not impose any additional requirement on the quantum mechanical scheme (other than linearity) for single measurements. The culprit then must be (M4).

Now, suppose we perform an ideal (preparatory) single measurement of A and subsequently we measure A and B jointly on the microsystems of a subensemble by a given pure state after the first measurement. Then the right-hand side of (M4a) may be interpreted as the joint probability distribution of A and B corresponding to this procedure as a whole. However (M4b) may be interpreted in an analogous way as the joint probability distribution of a joint measurement of A and B preceded by an (ideal) single measurement of B.

If as Muynck et al. propose, whenever $[A, B] \neq 0$ measurements of A and B are to be considered as totally different physical procedures, it seems we do not have much reason to expect that the two different procedures will give the same result. This provides a powerful insight into the fact that equality of (4a) and (4b) involves the commutativity of A and B; and that a joint measurement scheme incorporating this equality is bound to be inconsistent with joint measurement of non-commuting magnitudes.

Muynck et al., go on to explore requirements milder than the (M4) requirement. It is not important for us here to discuss and evaluate their proposed formal scheme. *What is important for our problem is to* *realize that a distinction must be made (or at least can be made) between a relation of non-disturbance and a relation of simultaneous (joint) measurement.*

If we assume that joint probability distributions are functions of the statistical state alone, as Park and Margenau, as well as Jordan (following Von Neumann) do, a purely mathematical definition of simultaneous (joint) measurement is possible. That is so because *if joint distributions are functions of state alone the experimental situation is not relevant for the joint measurement.* This is the key assumption that Muynck et al., have made explicit in their analysis of the concept of simultaneous measurability. It is this assumption that allows us to extend the unlimited precision of measurement in the case of single measurement to joint measurement. It is not surprising then that, as I show next, Jordan's requirement of unlimited precision implies Muynck's requirement of non-disturbance. I start by formulating both requirements explicitly.

Requirement D (Muynck's requirement). In the final state the relevant measurement instrument should have the corresponding pointer position with certainty if S is initially in a state described by an eigenfunction of A or B.

Requirement J (Jordan's requirement). If ϕ is a vector of length one for which $I_1\phi \neq 0$ then the probability that the quantities represented by A and B have values $\geqslant x$ and $\geqslant y$ (for any x, y real numbers) is $|I_1\phi| = |\phi^2| = 1$.

Now, if S is initially described by an eigenfunction ψ of A or B in a simultaneous measurement of A and B, then ψ (normalized) is a vector satisfying the condition for the vector ϕ given in requirement J. According to this requirement then the probability is 1 that the quantity A has value corresponding to eigenfunction ψ. Whence (assuming that the measurement is perfectly revealing) requirement J implies requirement D.

Thus a physical relation of simultaneous measurability relevant for an individual state interpretation of Luders' rule cannot be identified with T-compatibility, or at least, as it has been shown, this cannot be done in the strong sense presupposed by usual interpretations of Luders' rule. The physical interpretation of Luders' rule as a rule respecting a physical relation of simultaneous measurability can be based on the derivation sketched in Section 5, but the notion of minimal disturbance (or ideality) that plays a role in this argument is not independently grounded on a physical notion of (ideal) individual state transformation. The physical significance of s-ideality (see Definition 5.7 and Theorem 5.8) appears to be limited to a statistical interpretation of the Hilbert space metric.

7. CONCLUSION

It has been argued that an important distinction must be made between two problems of justification of Luders' rule. The justification of Luders' rule as a purely statistical rule is straightforward. Luders' rule can be proved to be the best statistical estimator of the final state conditional on the result of measurement. In that sense the rule is a purely mathematical result available to all interpretations. The problem of justification that remains is the problem of understanding the physical content of Luders' rule, if any, as a description of individual state transformations. The present investigation has disclosed a fundamental tension in the underpinnings of usual interpretations of Luders' rule. Luders' rule can be derived within the Hilbert space framework under the assumption that commutativity implies compatibility (Theorem 5.10), but the justification of the usual semantic interpretation of Luders' rule would seem to require also the converse direction, that compatibility implies commutativity. This tension, it seems to me, underlines the fact that usual interpretations are misled by Luders' converse type derivations in their attempt to grasp the physical content of Luders' rule. I have shown in this paper that usual derivations can only be seen as suggestions for a possible interpretation in terms of the idea of preserving or respecting a physical relation of compatibility. The most natural way of making sense of this idea is to think of Luders' rule as a state transformation that respects 'sharp-values'. As Herbut has shown this is sufficient for the derivation of Luders' rule. But we cannot be sure that 'sharp-values' have more than statistical significance. A usual implicit suggestion to overcome this difficulty is that Luders' rule can be seen as preserving a fundamental relation of simultaneous measurability which can be identified with T-compatibility. But I have shown that there are good arguments supporting the view that, at least in the strong sense required by the usual interpretation of Luders' rule, the physical relation of simultaneous measurability is stronger than T-compatibility. In this case the usual interpretation of Luders' rule as a transformation preserving sharp-values is inadequate.

I am not suggesting that Luders' rule has no physical content as a description of individual measurement transformations, but rather that the usual approach to this content is at best inconclusive and/or uninformative. In Martinez (1987) I suggest a quite different approach to the physical content of Luders' rule based on an altogether different type of derivation.

The above analysis also points the way to another serious difficulty of the usual interpretation of Luders' rule. The usual view takes Luders' rule as describing transformations that are subject to dynamical constraints (via simultaneous measurability for example). There are important arguments in the literature that suggest that these constraints are too strong, that there may not be any transformation satisfying the implicit constraints (see, for example, the paper by Stein and Shimony 1971). This problem, what can be called the problem of vacuity is a serious problem to the usual interpretation of Luders' rule. The problem of vacuity would only be more severe if the physical relation to be respected by Luders' rule is a weaker relation than T-compatibility (see Theorem 5.10).

It might be thought that I have left out of consideration an important alternative reading of the usual approach, the view that Luders' rule describes 'an approximation' which is 'ideal' in the sense that even though there are no actual individual state transformations described by it there are sequences of measurements that approximate Luders' rule with increasing accuracy. But for such a proposal to work a *physically significant criterion of minimal disturbance for individual transformations* must be given, and the arguments that I have provided against the 'literal' version of the usual interpretation can be made, mutatis mutandi, against the view that Luders' rule provides only a criterion of approximation. The suggestion that we take the metric (of the Hilbert space of operators) as a measure of an approximation for individual systems is only that, a suggestion, and I have shown that there is no obvious way in which such suggestions can be transformed into an argument.

NOTES

* This paper consists for the most part in material drawn from my dissertation (Martinez 1987) directed by Linda Wessels and Geoffrey Hellman.

[1] The reader might wonder whether Park and Margenau's theory of simultaneous measurement (Park and Margenau 1968) is relevant to the issue here discussed. But it is sufficient to point out that independently of any other consideration Park and Margenau's notion of simultaneous measurement, even if we were willing to accept questionable assumptions of their approach, would be irrelevant to the question of the physical interpretation of Luders' rule as a description of individual state transformations. Park and Margenau's notion of simultaneous measurability, as this notion is implicit in examples of the time-of-flight method, is at best approximate for any finite time t.

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