Mathematical Pictures

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[draft - comments very welcome]

Abstract: In this text I develop the thesis that geometrical diagrams are depictions, not symbols; they depict geometrical objects, concepts or states of affairs. Besides developing this claim, I will defend it against three recent challenges from (Sherry 2009), Macbeth (2009, 2010, 2014) and (Panza 2012). First, according to Sherry, diagrams are not depictions, because no single depict can depict more than one thing, yet a single geometrical diagram can represent different geometrical figures in different contexts. I will argue that, once we recognize that resemblance underdetermines depiction, we can see that pictures can indeed depict different thing in different contexts and, consequently, there is nothing surprising about a single diagram depicting different geometrical figures. Next, I will defend it against a similar argument by Macbeth, according to which diagrams can represent different geometrical figures, even within the context of a single geometrical proof. Finally, I will defend it against Panza's argument that there are essentially spatial features that geometrical objects have only insofar as they inherit them from the diagrams that represent them, and this is incompatible with the hypothesis that geometrical diagrams are depictions for depictions inherit their visual and spatial properties from the objects they represent, and not the other way around. In response, I will argue that once we understand the sense in which subjects are metaphysically prior to their depiction, we will see that my claim that Euclidean diagrams are depictions is not incompatible with Panza's thesis.

1. Introduction

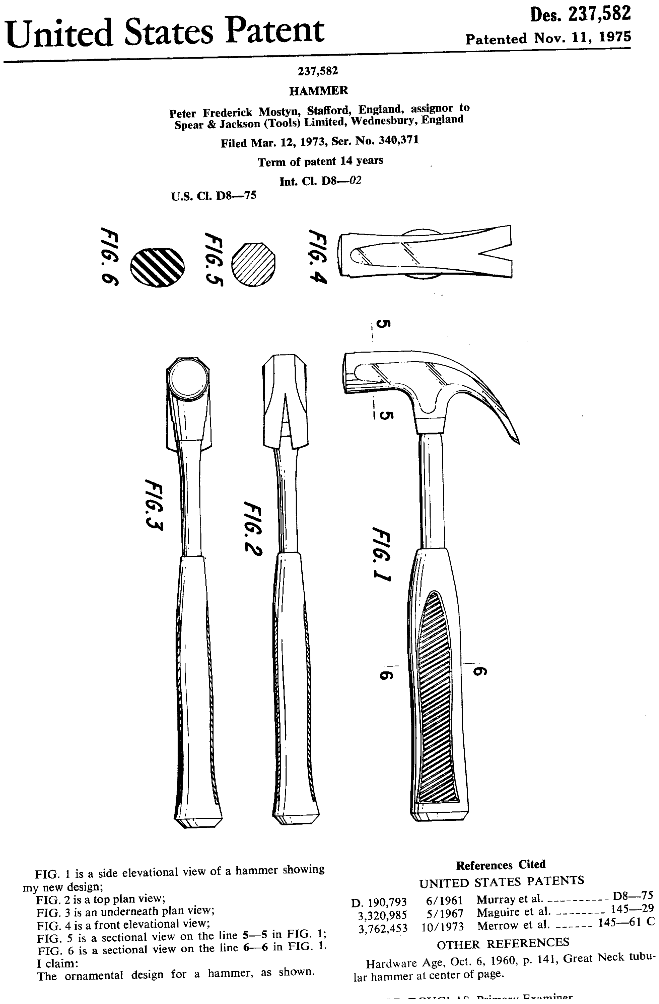
Diagrams play a central role in many aspects of current mathematical practice – in education, popularization, discovery, application, etc. (Mancosu 2005) Today, mathematical concepts and propositions are commonly presented in diagrammatic form, both in classrooms and research seminars all over the world. In contrast, the rigorous study of mathematical diagrams has been a mostly neglected topic, both in the history and the philosophy of mathematics. The traditional philosophical stance towards diagrams in mathematics is simple and well-known: Proofs that make substantial use of diagrams are assumed to lack the kind of rigor necessary to ground mathematical knowledge and are therefore of no interest to philosophy (Borwein 2008). As Neil Tennant has famously asserted, "It is now commonplace to observe that the diagram is only an heuristic to prompt certain trains of inference; thatit is dispensable as a proof-theoretic device; indeed, that it has no proper place in the proof as such. For the proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array." (Tennnat 2006)

In recent years, this stance has been challenged by a growing body of work aimed at reevaluating the role of diagrams in mathematics, its history and philosophy (Brown 2008, Guiaquinto 2007, Lomas 2002, etc.). Some of this work (Krummheuer 2009, Sherry 2009, Kulpa 2009, Brown 2008, Norman 2006, Shimojima 1996, Barwise 1993, etc.) has tried to show how diagrams actually play or have played a substantial role in the justification of mathematical knowledge, and that diagramatic proofs can be as rigorous as formal ones. Others (Corfield 2003, Mancosu 2008, Van Kerkhove & Van Bendegem 2007, etc.) have advocated for an alternative agenda for the philosophy of mathematics, one that does not focus on an abstract and idealized conception of mathematical justification but instead deals with actual mathematical practice and the central role diagrams play in it.

Despite this recent increase in philosophical interest in mathematical diagrams, fundamental questions still remain: ¿What is their epistemic value, if any?; ¿What role do they play in mathematical justification, explanation or understanding?; ¿Do they represent? and if so ¿what can they represent and how?; ¿Do they have a syntax?, ¿a semantics? ¿a pragmatics?; ¿Is their representational character mediated by resemblance, like pictures, or by convention, like symbols? ¿How have techniques and styles of diagrammation evolved and why? In general ¿what theoretical challenges emerge from what Inglis and Mejía-Ramos have called "the ubiquity of visual argumentation in mathematics" (2009)? The aim of this article is to contribute to answering one of these fundamental questions: *are geometrical diagrams depictions?*

2. Representations as Tools

When we use a representation to perform an inference – as when we use a diagram to prove a geometrical result –, we are effectively using the representation as tool to perform a task. As such, it might be fruitful to frame our questions regarding geometrical diagrams within the framework of tools and their philosophy. In this respect, representations are not much different from simple tools like hammers, tractors or lamps. To explain each one of these tools, that is, to explain why they exist, and why they are the way they are, one must relate them to their function, i.e. to the role they play in our performing of particular tasks they help us perform. In other words, it is impossible to explain tractors without talking about plowing and agriculture. In a similar fashion, to explain why hammers are the way they are, it is impossible not to take their function into account. It would be impossible to explain why a hammer is shaped as it is, or why it is not fluffier and made of cardboard without mentioning what it is for. Consider the hammer's handle. Why are they as long as they are? Why not shorter? The explanation is simple, once we realize that hammers, when used to drive in nails, work as levers, third-class levers. In levers of this kind, the effort is placed between the load and the fulcrum. In the hammer, the wrist serves as fulcrum, while the effort is applied through the hand, and the load is the resistance of the wood. (*World Book Encyclopaedia* 1979) In third class-levers, the effort must travel a shorter distance and must be greater than the load. Thus, a handle of a shorter length would not potentialise the effort enough to make the hammer efficacious.


United States Patent

Des. 237,582
Patented Nov. 11, 1975

237,582
HAMMER
Peter Frederick Mostyn, Stafford, England, assignor to Spear & Jackson (Tools) Limited, Wednesbury, England
Filed Mar. 12, 1973, Ser. No. 340,371
Term of patent 14 years
Int. Cl. D8—02
U.S. Cl. D8—75

FIG. 1 is a side elevational view of a hammer showing my new design;
FIG. 2 is a top plan view;
FIG. 3 is an underneath plan view;
FIG. 4 is a front elevational view;
FIG. 5 is a sectional view on the line **5—5** in FIG. 1;
FIG. 6 is a sectional view on the line **6—6** in FIG. 1.
I claim:
The ornamental design for a hammer, as shown.

References Cited

UNITED STATES PATENTS

D. 190,793	6/1961	Murray et al.	D8—75
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Hardware Age, Oct. 6, 1960, p. 141, Great Neck tubular hammer at center of page.

But not only that, it is also almost impossible to explain why our tools are the way they are independently of how we are ourselves. It is a truism that a hammer for aliens might not be similar to a human hammer. Why are the bases of hammer handles ovals or similar and not, say, square or rectangular? Because square handles would be uncomfortable for human hands, while oval handles provide optimum comfort and avoid stress and injury. A hammer with a square handle would be as efficacious for driving in nails as hammers with oval handles. However, they would be less ergonomic. In general, when explaining features of our tools, sometimes it is not enough to appeal to what the tool is for, but also how the *user* is: It makes no sense trying to explain why hammers have the dimensions they have without referring to the size of our arms and hands. Even if aliens developed hammers, it is very unlikely that they would be similar in shape to ours if aliens were not also similar in shape and physiology to us.[1] We can explain some features by explaining how the task constraints the kind of object that can perform such task, but also the user constraints the tools in similar ways. Notice that this distinction is a distinction in factors that shape our technology, not primarily of features. Some factors might weight more in the explanation of some features, but most likely, features will be the result of the influence of both factors.

Summarizing, regarding tools, it is impossible to account for their form without mention of function. Furthermore, it is also impossible to explain why our tools are the way they are independently of how *we are ourselves*. In general, technology is strongly shaped, among other factors, by constraints on how users are and what they can and cannot do as much as what goals such users wish to achieve through their use of such tools. So far, this must not be controversial.

Representations are as much tools as hammers. They are also devices used to assist us in performing certain tasks. Representations help us communicate, but also help us understand and navigate the world, make art and wage war, work and play, etc. Almost every human activity exploits some form of representation to make our tasks easier, simpler … or more fun, more beautiful, etc. As such, out theories of representation must not be much different from our theories of other human tools. In particular, in explaining why our representations are the way they are, we must take in account what we develop them for, as well as how they accommodate our human condition, our strengths and weaknesses. Just as it is a truism that a hammer for aliens might not be similar to a human hammer, so alien representations might not be similar to human representations.

Some people might object to my talk of the many uses of representation *in plural.* After all, it might seem that for something to be a representation, it must serve one fundamental function, i.e., to *represent* something. All representation are used to represent, and anything that is used with a different purpose is thus no longer a representation (This seems to be the position of David Sherry 2009, Valeria Giardino 2012 and Tarja Knuuttila 2011). The point is well taken, and so when talking about the uses

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[1]. Of course, I am not claiming that these are the only two factors shaping a technology. However, it is these two kind of factors that will be important for our goals ahead.

of representations one must consider both their narrow function, namely to *represent*, and also the multiple wider purposes they serve in our lives. For example, it is true that the signs outside sex segregated public restrooms have the function of representing a man and a woman; but it must also be obvious that they are there with the purpose of helping users identify which restroom is assigned to which sex. The first is their narrow representational function, while the second is their wider purpose. Most if not all representations have a wider purpose besides their narrow representational function. Riffing on Austin's dictum, there are many things we do with representations.

The importance of considering the wider purpose of representations, instead of focusing only on their narrow representational function must be obvious once we think of some examples. For example, it must be fairly uncontroversial to claim that any account of representational artworks that deals only with their representational function while ignoring their artistic purpose would be severely shortsighted. Something similar can be said of the images of political propaganda: they are shaped not only by their narrow representational function, but also by their wider political purposes. Failing to take both into account would leave us with a very limited picture of their nature.

Summarizing, to understand why human representations are the way they are, we must understand not only what they aim to represent, but also why we might want them to represent that and in what circumstances. Furthermore, we must remember that in order to fulfill their narrow representational function, representations must play a communicative role that essentially assigns at least two roles to their users: transmitter and receiver. In a sense, both are users of the representation as tool, and successful representations must meet the needs of both in an efficient way, that is, without spending too much of their resources. So, continuing with our previous example, in order to explain why the aforementioned restrooms signs are the way they are, we might sometimes need to appeal to their narrow representational function of representing a man and a woman, and other times to their wider purpose of helping identify restrooms for men and restrooms for women; we might sometimes need to appeal to how they fit the needs, limitations, etc. of potential restrooms users who can benefit from the information contained in the signs, and other times to those of whoever wants to communicate that information to them. For example, we might explain why the figures in the women's restrooms signs are wearing a skirt by appealing to how, even though neither all women wear skirts, nor everyone who wears a skirt is a woman, we (possible restroom users) *expect* skirt wearers to be women, and thus we (who want to help these potential users identify the sex corresponding to each restroom) can exploit this expectation to signal a sexual difference between these figures and the figures in the other restroom.

I do not expect these simple examples to completely illustrate the importance of distinguishing between a representation's narrow representational function and its wider purpose of use, and between the way these function and purpose shape them and how the circumstances of their users also determine their form. On the contrary, the main goal of this chapter is to illustrate this by developing a couple of examples in fuller detail and show how making these distinctions allow us to clarify certain debates in the literature. In particular, I will deal with a particular subset of representations that have attracted a lot of philosophical interest in the last few decades, specially in the philosophy of science: those that are used as inferential tools, that is, tools that aid us in making inferences. In what follows, I will try to show that, when dealing with representations of this kind, it can be useful to distinguish between the issue of how our inferential goals shape the representations we use, and the question of how such representations complement, exploit, and extend our cognitive abilities. In other words, when dealing with successful inferential representations, the questions of why we use them entails two different but closely related questions: first, why it is advantageous for us to use them, and second, why are we justified in using them. We must ask both how well the representation fits the task, and how well it fits us; both how efficient it is to use the representation, and how efficacious it is. From now on, I will call questions about how our constitution as users shape our inferential representations "ergonomic" to distinguish them from more "logical" or "epistemological" questions about their inferential efficacy. Needless to say, philosophers have focused almost exclusively on these later questions and have thus ignored an important dimension of our inferential representations.

3. Inference and Representation

In 2007, in one of the most dramatic moments in winter sports, french biathlete Raphaël Poirée took to the mass start race at the Holmenkollen World Cup for what he had publicly announced to be the last competition of his illustrious career. However, as he crossed the finnish line, his long time competitor Ole Einar Bjørndalen of Norway was still right by his side. Fortunately, the finish-line camera had photograph the race's final moments (Figure 1). From these photos, the jury determined that Poirée could not finish his career with a winning race, for Einar Bjøorndalen had crossed the line just instants before.

Besides its dramatic nature, the aforementioned episode is of interest for us philosophers because it illustrates the central role representations like photographs play in the obtaining of new knowledge. When dealing with episodes like this, the philosopher is interested in determining what epistemic advantages do we glean from the use of representations, and why are we justified in doing so. In other words, why, in drawing certain conclusions about one give situation, instead of probing the situation itself, we make use of representations of it, and second, when we do, why are we justified in accepting the conclusions we reach through them. For example, why did we need to use a photograph to the determine who won the race, when it occurred just right in front of our very own eyes, and second, why were we justified in accepting the conclusion we thus reached. In general, when talking about the successful role of representations in knowledge, the same two sort of questions arise. Let me call the first kind of questions, logical; and the second ones, cognitive. These different sorts of questions require different sorts of answers, answering the logical question regarding the use of photo-finish might require saying something about the causal process behind photography and maybe also something about the location of the cameras in relation to the finish line. In contrast, answering the second, cognitive question might require saying also something about the limits of our perceptual system and thus why we could not see the winner with the naked eye, etc.

Fig 1 Photo Finish

In order to understand just what might be involved in answering each sort of question, it might be useful to look into what happened that day in Holmenkollen in more detail, to see a general pattern that we might be able to generalise to other inferential uses of representations. The starting point is a situation from which we want to get certain information, in this case, the ski race. There is something we want to know about it, i.e., who crossed the line first. In order to answer this question, we proceed by first producing a representation of the situation: one that represents not the whole of the situation at hand, but (at least some of) its relevant aspects in a tractable way. In other words, we want a representation that carries enough relevant information about its subject, without adding too much noise, that is, without including elements that might be mistaken as representing something relevant about its subject without actually doing so. Once we have such a representation, we translate our original question about the target situation into a corresponding question about its representation. In our example, we translate the question of who crossed the line first to a question about features of the photograph and what they represent, i.e. to the question of whether the part of the image representing the foot of one skier touches the part of the photo that represents the crossing line, while the part representing the foot of the other skier does not. If the system works properly, then the answer we get to this second question will serve us also to answer the original question, once properly translated (Barwise 1993, Suárez 2004).

Notice how in drawing on the picture to get information about the race, we perform several inferences, and not all on a par. In broad terms, we an identify two different – and equally important – sorts of inferences: inferences of the first sort go from features of the picture to what the picture represents, while inferences of the second sort go from what the picture represents to how the world is. In the first sort of inferences, we go from information about the picture – mostly, about how it looks, but also background information about it – into information about the representation's content, i.e., about how the world is *according to* the representation. We infer from the pattern of colours and shapes shown in the photograph that, in the picture, the racer in red crossed the line later than the racer in black. And, if we take into account more information about when and how the photo was taken, about who was racing and how they looked like, etc. We also arrive to the conclusion that, *in the picture,* Bjøorndalen crossed the finish line before Poirée, that the picture represents Bjøorndalen beating Poirée in Holmenkollen, etc. Inferences of this sort constitute what is commonly called the *interpretation* of the representation.

Once we have performed these inferences and have determined what is represented in the picture; we still need to proceed to inferences of the second sort. Once we know how the world is according to the picture, it is important to determine whether the world is actually as the picture represents it to be. Once again, background information about the representation and how it was created will be essential. If we trust the process of photo-finsh and this particular instance, we can conclude that Bjøorndalen crossed the finish line before Poirée, not only according to the picture, but *in the real world.* We conclude that the picture not only represents Bjøorndalen beating Poirée in Holmenkollen, but that it *shows* it.

Sometimes, in order to extract from a representation the information we need, it might be enough to just inspect the representation, looking for the relevant information in it. However, other times, the solution might not be so straightforward, and some manipulation of the representation might be required to extract the relevant information. For example, in photo finish situations, it is common practice to draw parallel lines on top of the photographs marking the edge of each racer closest to the finish line. In this process, it might be also necessary to combine the information contained in the representation with background knowledge from the target situation itself. When using a map to navigate a city, for example, it is commonly necessary to match information from the map and information available at the context of use to determine what route to take or even to identify just where one is. In other words, sometimes, there is a going back and forth between representation and target in order to interpret the representation, and thus acquire the desired information.

Most of the times, when we use a representation to perform an inference, we need to proceed through these two sorts of inferences. We need to first interpret the representation, before we can infer something from it about the world. After all, information contained in a representation is useless if it is not extracted from it, and applied to the world. Both elements are fundamental. Different sorts of representations require different sorts or interpretation, and different representations are trusted for different reasons. When we see the light flashing through the numbers inside an elevator, for example, we trust the information it gives us about the elevator's itinerary for different sort of reasons that why we trust a photograph we took ourselves, or the maps on a subway station. Similarly, how we interpret a text is significantly different to how we interpret a map or how a radiologist interprets an X-Ray image. When talking about images, diversity is the norm.

In every case, the whole process involved in using a representation to perform an inference is rather complex: it involves, not only the generation of the representation of the target situation, but also the interpretation, manipulation and evaluation of the representation. All this to draw an inference that, at least in principle, could have also been achieved working directly on the world. This means that it makes sense to use a representation to draw an inference, only when its use has some advantage, either in perspicuity, certainty, accessibility, etc. over working directly on the subject of such inference. This means that what makes a representation good for a certain inference, is not just its accuracy in representing its target or its reliability in producing valid inferences, but also its usefulness: its tractability, accessibility, clarity, etc. As I have mentioned, a good inferential representation is one that helps *us* reach *our* inferential ends; as such, it must help us overcome our limitations and/or capitalize on our capacities in reaching our inferential ends. In other words, it must not only be effective in giving us the information we need, but must do so in an efficient way (Giardino 2012, Kulvicki 2010, Blackwell 2008,). Sometimes, the advantage is that the representation permits us *see* what we could not se otherwise. The photograph in the photo finish example above, for example, allows us to see something – the race's last instant – that we could not otherwise see, even though it happened right in front of our very eyes. In this sense, it extends the capacities of our very eyes.

In general, the above example of the photo finish clearly illustrates how both ergonomic and logical constraints shape our representations, specially representations used in inference, where ergonomic and logical constraints shape both how we interpret the representation and whether we trust the representation to tell us something about the actual world (instead of the world as represented in the representation).

4. Geometrical Diagrams as Inferential Representations

Let me turn now to a different example of inferential representation: geometrical diagrams. In the above regards, geometrical diagrams are not much different from other representations used to draw inferences. We use them to make inferences about the geometrical realm that it would be hard, if not impossible, to make just thinking directly about geometrical objects (Dutilh Novaes 2013, Macbeth

2014). We start from certain information given about a geometrical situation and construct a diagram to help us draw inferences about it. Sometimes, in order to get the result we want, we need only look at the diagrammatic representation of our given information, but most times we need to transform the diagram until the desired result is displayed (Hintikka & Remes 1974, Mumma 2012). Once displayed in the diagram, the result must be translated back into a general geometrical claim that is no longer about the diagram, but about the target geometrical situation. In this sense, diagrams are not very different from other representations used to draw inferences.

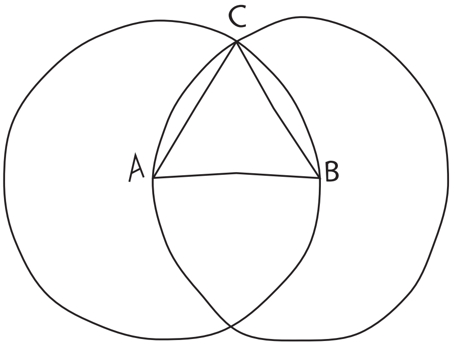

Fig 2 Euclid I.1

Let's look at a simple example: the famous diagram used in the proof of Euclid I.1. that requires the construction of an equilateral triangle on a given straight line (Figure 2). The first step is to diagrammatically represent the given line. This is done so by drawing a short, and more-or-less straight horizontal line. Why do we do this? One might think that it is easy to see why the line has to be more-or-less straight, but why horizontal? and why short? It is clear that we could have made the line shorter or longer; vertical, horizontal or at an angle, and the logical validity of the proof would not have changed (Manders 2008, Mumma 2010). However, we do not. We know that if we drew the line too short or too long, the diagrams would loose, not validity, but usefulness. We need the diagram to be tractable and that adds extra restrictions about how better to represent, say, a straight line.

Once the line is in place, in order to identify it, we use two indexes. We write a letter "A" at one of its edges and a "B" on the other. We use these indexes to identify the edge points of the line, and the line itself in the text of the proof. These indexes are essential to the interaction between diagram and text (Netz 1998). Since the goal is to draw an equilateral triangle, whose sides are the same length as the

AB line just given, we know that one of the aspects that will be relevant about line AB will be its length.; but at this stage of the Elements, we know very little about the length of lines. We know, for example, that all the radiuses of a circle are of the same length. And also, we know that, given a straight line, we can construct a circle that has such line as its radius. Thus we can use this information to enrich our diagram. We do so by drawing a more-or-less circular curve in such a way that edge point A lays more-or-less at its center and edge point lays on its circumference. Just like it is not necessary to draw a perfectly straight line to represent one, so we do not need to draw a perfect circle to represent one. It just has to be similar enough. And we can do the same thing on the other edge. As is well known the resulting curves will intersect at two points, one over the line AB and another one below. Take one of these points. Once again, it makes no logical difference which one one takes, but we usually take the one on top. Why? Because we prefer our triangles pointed upward. (Friedenberg 2012) The difference is not logical, but ergonomic (Figure 3).

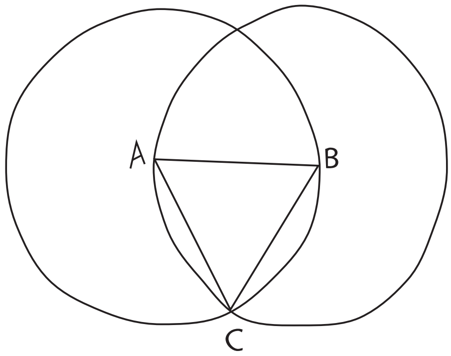

Fig 3 Inverse Euclid I.1.

So, we take that point, label it with a letter "C" and draw a more or less straight line from it to point A an another line from it to point B. What results looks more or less like a triangle, and it represents one. How do we know that is is equilateral? Not because we can see that its sides are more or less the same length (Manders 2008), since our capacities of determining objective and relative length are known to be very deficient and unstable (Sedgwick 1986, Gogel 1990). Instead we know this because of a proof. And this proof is performed as much on the page, as on the diagram. We know that line AB and BC are of the same length, not only because of how we see them in the diagram, but because they are both radiuses of the same circle. We know that AB is a radius of the circle with center on A, because that is how we constructed the circle, and we know that AC is also a radius of this same circle because that is how we constructed point C. A symmetrical line of reasoning leads us to the knowledge that CB is also of the same length as AB. By transitivity, we get that AB, BC and AC are all of the same length and consequently, triangle ABC is equilateral.

Notice how what happens in the diagram here is completely analogous to what happens in the photo finish case. How do we know of certain portion of the photograph that it represents one or other of the racers? Partly it is because of how it looks. We can see the red on the photograph is similar to the red on Bjøorndalen's uniform, for example, and this guides us in identifying its referent. However, if we did not know anything about how the photograph was taken, i.e., how was it located, when was the picture taken, etc. we would not be able to fix the referents to the shapes we see on its surface. We know that the red figure is Bjøorndalen because we know that the norwegian racer wore red to the race and because we know which race was being photographed, etc. In other words, in photographs – and in diagrams, I will argue –, visual resemblance restricts content, but it does not fully determines it. In other to fix a referents to a depiction, we need rich contextual information, both about what is being represented and about how the representation was constructed.

If I am right, a proper philosophy of diagrammatical reasoning in geometry must address not only the question of why (and when) are geometrical diagrams reliable means for making inferences about the geometrical realm (Mumma 2010, Krummheuer 2009, Kulpa 2009, Brown 2008, Guiaquinto 2007, Lomas 2002, Norman 2006, Shimojima 1996, etc.), but also why they are useful for doing so.

Before moving on, let me give a further example to to illustrate the difference between these two sorts of questions. Consider the diagrams in figure 4 representing the logical relations between three sets:

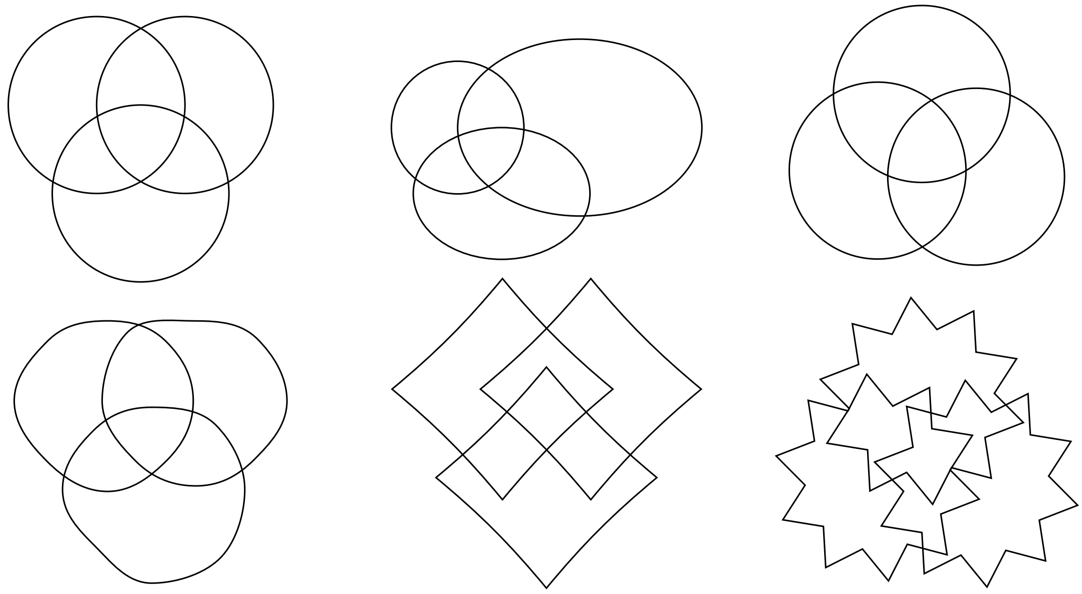

Fig 4 Isomorphic Venn Diagrams

Even though they all share the relevant topographical properties necessary to reliably represent their target, i.e., the logical relations between sets (Shimojima 1996), some of them are better suited to this task than others. Some are clearer, some are more confusing; some are easier to draw, others more complex; some seem more beautiful, others are ugly. And these differences are not logical, but of a different kind. These are the kind of differences I am calling, following Blackwell, "ergonomic" for they depend not only on the internal properties of diagrams themselves, but also of the capacities of their users (and other external factors, such as the material restrictions associated with the practices within which they are used).

This means that we must expect that what makes a geometric diagram helpful for a given proof be not its accuracy in representing a geometrical object or state of affairs, but also its tractability, clarity, etc. In what follows I will argue that partly, why we use diagrams in geometrical proof is similar to why we use photographs to decide who won a race, i.e., because diagrams, like photographs, look like what they represent. As I mentioned before, in appropriately using a representation to make an inference, it is important to be able to identify what is being represented. Thus, all things being equal, it is desirable that representations be developed in such a way that their referents are easy to identify (Paraboni et al. 2007). Different kinds of representations use different mechanisms to determine their referents and to make them easily identifiable. One of the most common mechanisms of reference fixing is the establishment of a convention through some kind of "baptism" (Kripke 1980), but there are others. For example, most scientific models ground their semantic relation to the world on being similar to what they represent (Giere 1988, 2004, Teller 2001). This similarity can be structural or in appearance, i.e., the representation might either be homomorphic to its subject or it might resemble it (or a combination of the two). When the semantic links is at least partially grounded on resemblance, the representation is called a *depiction*. Audio recordings, realistic drawings and sculptures, etc. are all examples of depictions.[2]

Depictions have a clear cognitive advantage over other kinds of representations: when determining what something represents, it helps a lot if the representation looks similar to its referent. If we look back at the photo finish example above, we will see that even though there could be other mechanisms that could accurately report the information of who crossed the line first, the photo finish has become a standard mechanism partly because among its practical and cognitive advantages, photographs look like what they represent. What we see when we see a photo finish is pretty similar to what we would have seen if we could have seen the final instant of the race frozen in time in front of us. This makes the information the picture contains about the race easily accessible, and its reliability very vivid. This means that part of why we epistemically use photographs in cases like this is precisely because they look like what they represent. In what follows I will try to argue that this later claim is also true about geometrical diagrams, i.e., they also look like what they represent and, furthermore, that is partially why they succeed in representing the geometrical objects they do.

It is commonly assumed that geometrical diagrams are substantially different from words and mathematical formulas; that diagrams are pictures, while formulas and words are symbols, and that therefore, mathematics conducted diagrammatically is substantially different from mathematics conducted through formulas and words. The main hypothesis I will try to defend here is that, at least in

2. As with any substantial philosophical thesis, there are some who challenge the claim that depiction is grounded on resemblance (see for example Goodman 1968; Wollheim 1998; Lopes 1996; Greenberg forthcoming; etc. Strong defenses of the thesis that depiction is grounded in resemblance can be found in Abell 2005, 2009; Hyman 2006; Hopkins 1994; Peacocke 1992). Even though I am convinced that the resemblance theory I will present here can meet the challenges raised by these authors, I will not defend this view here.

the case of purely geometrical diagrams,[3] our commonsensical intuition is right: geometrical diagrams are pictures, not symbols; they depict geometrical objects, concepts or states of affairs.[4] I do not mean to claim that geometrical diagrams are the only cases of depictions in mathematics, but they serve as paradigmatic examples of mathematical depictions. Since this is a universal claim, it is almost impossible to demonstrate directly. Instead, I am going to offer what I hope is a convincing account of how geometrical diagrams depict geometrical objects, give a few examples, and then address some qualms my account might raise. Specifically, I am going to defend my hypothesis against alleged counterexamples by David Sherry (2009) and Danielle Macbeth (2014), according to whom geometrical diagrams can be successfully used to prove theorems about geometrical objects and situations they do not resemble and thus cannot be pictures of, and a second argument against it by Marco Panza (2012), according to whom, geometrical diagrams cannot be depictions, for at least some of them are metaphysically prior to the geometrical objects they purport to represent.

The articles is structured is as follows: First, I will present and develop the hypothesis that geometrical diagrams are what in the recent literature on visual representation have been called *depictions*. The portrait of Benjamin Franklin on the American hundred dollar bill, Jeff Koon's porcelain sculpture of Michael Jackson and his pet chimpanzee Bubbles, and the embroidered crocodile monotype of La Chemise Lacoste on the chest of my sweatshirt are all uncontroversial examples of depictions. It is equally uncontroversial that abstract artworks like Mondrian's *Compositions*, written words like the ones that constitute this text, and logos like those of Toyota and Citroën are not depictions. I will work under the assumption, developed by authors such as Hopkins (1994), Hyman (2006), Peacocke (1992) and Abell (2005, 2009), that depictions, unlike other visual representations,

3. The adjective "purely" is meant to exclude diagrams of differential or algebraic geometry like those developed by David Mumford (1999). From now on, when I talk of geometrical diagrams I will mean purely geometrical diagrams in this sense.

4. This idea dates at least as far back as Plato's account of geometrical diagrams in Republic VII, 527a6-b6, where he writes: "when mathematicians are doing geometry, describing circles, constructing triangles, producing straight lines, they are not really creating these items, but only drawing pictures of them". Cf. Taisbak 2003 and Panza 2012. Notice that this assumption does not identify depiction with visual resemblance. Matters are a little bit more complex than that, so resemblance is a necessary but by no means sufficient condition for depiction. Determining what else is required for depiction will be part of what I will do throughout the paper.

represent partially in virtue of their visual resemblance to what they represent.[5] In order to make my hypothesis plausible, I will also need say something about the putative abstract nature of geometrical entities and how this is compatible with my assumption that they have sensible properties nevertheless. Then I will present Sherry and Macbeth's puzzle, i.e. how is it possible for a single diagram to represent different incompatible objects? I will argue that, once we understand what it takes for an image to depict an object, the puzzle dissolves. In particular, I will claim that, once we understand that resemblance is a necessary but not sufficient condition for depiction, we can easily see how the same picture can be used to depict different objects in different contexts. Next, I will present Panza's challenge, i.e. that my hypothesis that Euclidian diagrams are depictions entails that their visual features are metaphysically dependent on those of their subject, and this is incompatible with the broadly Aristotelean thesis, defended by Panza among others, that at least some geometrical "objects inherit some properties and relations from these diagrams" (Panza 2012, 55). To meet this second challenge, I will show how my hypothesis does not actually entail the offending claim, and so is compatible with Panza's brand of Aristotelianism. Finally, I will draw some general conclusions about mathematical diagrams and raise further questions.

5. On Depiction

The main claim I want to defend in this article is that geometrical diagrams are depictions of geometrical objects, i.e. they represent whatever geometrical figures they stand for partially in virtue of visually resembling them.[6] To better understand what it means for something to depict, it is a good idea to contrast depiction with other ways an object can represent another. Symbols, for example, usually represent their referents by convention, not depiction. Take words; in general, they do not depict their

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5. As with any substantial philosophical thesis, there are some who challenge the claim that depiction is grounded on resemblance (see for example Goodman 1968; Wollheim 1998; Lopes 1996; Greenberg forthcoming; etc.). Even though I am convinced that the resemblance theory I will present here can meet the challenges raised by these authors, I will not defend this view here.

6. From now on, when I talk about the representation/depiction of objects, I will mean to include also the representation/depiction of concepts and states of affairs as well. It is important not to assume that all depictions are of objects (see Blumson 2009).

meanings, because they seldom resemble the things they represent. The English word "albatross" does not look anything like an albatross, just like "lima bean" does not resemble a lima bean; the IEC standard standby symbol used on many home appliances does not resemble the standby state, just like the Chase Manhattan bank's logo does not resemble a financial institution.

Fig 5. The World Wildlife Foundation Logo

Most pictures, on the other hand, resemble what they are used to represent. A picture of an albatross commonly looks like an albatross in ways that the English word "albatross" does not. When pictures represent the objects they do, partly because they visually resemble them, we say that they *depict* them. In this sense, Chuck Close's gigantic portrait of Phillip Glass depicts Phillip Glass, just as Imogen Cunningham's floral photos depict callas, magnolias and other flowers. Thus understood, depiction is foremost a *relation,* and only derivatively a *kind of object* (an object that depicts). Consequently, a depiction can also be a symbol if, besides depicting something, it symbolizes something else. Non-figurative pictures like those of abstract, conceptual and similar avant-garde art are eminent examples of this, i.e., object that depict something, but symbolize something else (Abell 1995). Outside the world of fine arts, logos, coats of arms, and other visual symbols are also used to represent more than they depict. The logo of the World Wildlife Foundation (figure 5), for example, contains the picture of a panda bear. The picture depicts a panda, but the logo does not. It represents a foundation dedicated to defending pandas and other endangered animal species. In this case, there is a conventionalized metonymic relation between what the picture depicts (a panda) and what it represents (a foundation). In cases like these, we say that the picture depicts one thing, but symbolizes another

(Arnheim 1969). This commonsensical distinction is commonly made in ordinary language by distinguishing what the picture is a picture *of* and what else it represents or symbolizes.[7] That is why a picture of a bald eagle, to use another example, can represent the United States of America, without being a picture of that country. A picture of a bald eagle can represent the U.S.A., not by *depiction* (the mechanism linking picture and country is not grounded on visual resemblance), but because of an explicit convention, in effect, an act of the U.S.A. Congress from June 20, 1782 (Atwood 1990). Now, when I claim that geometrical diagrams depict geometrical objects, I mean to say that their relationship with the mathematical entities they represent is like the one between a picture of a bald eagle and a bald eagle and not like that between a picture of a bald eagle and the United States of America. Thus, my claim is not just that diagrams represent geometrical objects, but that diagrams *depict* what they represent.

Now that we know what it means to say that something depicts or is a picture of something else, we can turn our attention to mathematical diagrams. The first thing to notice about mathematical diagrams is that they do not form a homogenous natural kind with sharp boundaries.[8] There is no sharp line between what is and what is not a diagram. Take two-dimensional arrays like tables (Figure 6), matrices (Figure 7), or the formulas on Frege's *Begriffschrift* (Figure 8). Are they diagrams or are they formulas? Some may argue that, since they exploit spatial relations for representational purposes, they must count as diagrams (Giardino 2012). Others might argue that since they follow strict syntactical rules and their semantic is compositional, they must instead count as symbols (Kahl 2003).

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7. Notice however, that "depicting" and "being a picture of" may not be perfect synonyms, since we may still say that something is a picture of something it fails to depict it. For example, according to Kaplan (1968), a blurry photograph of Mount Rushmore is still a picture of Mount Rushmore, even if it does not depict Mount Rushmore, i.e., it does not resemble Mount Rushmore in a way that it is rationally to expect an audience to recognize. Also, not everything we call a picture is a picture *of* something. Abstract pictures, for example, are commonly considered not to be pictures of anything, even if they (or at least some of them) manage to represent. Consequently, abstract pictures do not depict. I thank Alex Grzankowski for raising this issue.

8. When I talk of mathematical diagrams, I will be talking not about abstract diagram types, but about concrete diagram tokens. I will also assume that (at least some) geometrical diagrams are geometrical figures themselves.

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Fig 6. Multiplication Table

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Fig 7. Adjacency matrix for graph K_5

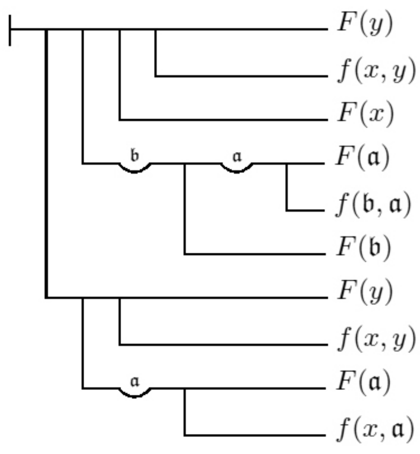

Fig 8. Frege's Begriffschrift Theorem 71

Something similar can be said about other sorts of mathematical diagrams, like the lines and dots in graphs (figure 9), the arrows of category theory (figure 10) or the circles in a Venn Diagram (figure 11). They are neither symbols in a traditional formal language, nor depictions of anything. I have very little of interest to say about diagrams of these kinds. Consequently, from now on and except where explicitly stated, when I talk about diagrams, I will mean geometrical diagrams.

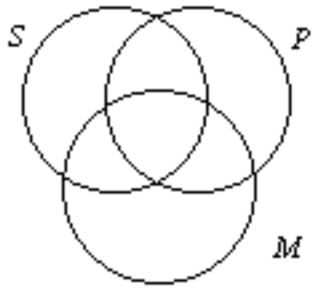

Fig 9. Venn Diagram

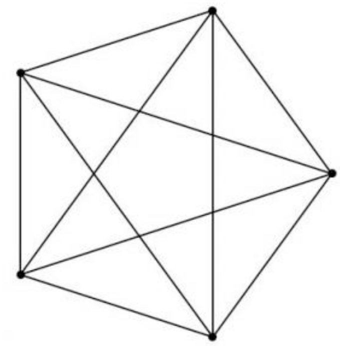

Fig 10. 5 vertices graph (K_5)

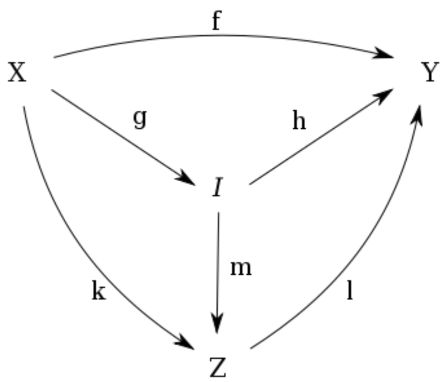

Fig 11. Diagram of Category Theory

Among the mathematical diagrams that contain pictorial elements, in contrast, we have geometrical diagrams like those in figures 8, 9 and 11. These are the diagrams I aim to cover in my hypothesis. I do not mean to claim that geometrical diagrams are the only cases of depictions in mathematics. My intuition is that, besides geometrical diagrams, some arithmetical diagrams like those used in so-called visual proofs (figure 12) are also depictive, but I will not defend this hypothesis here.

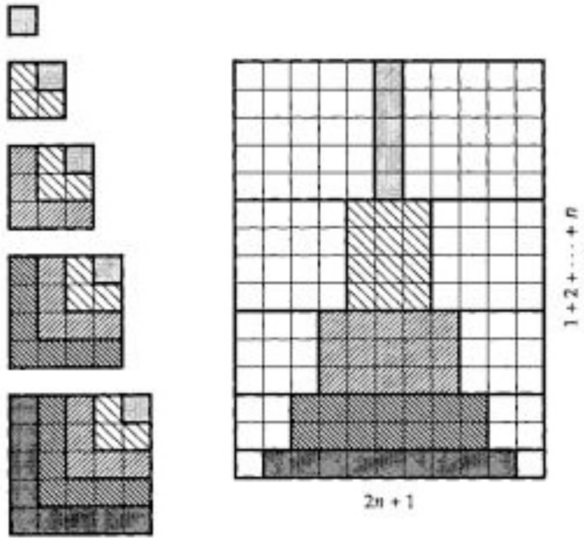

Fig 12. Arithmetical visual proof (Nelsen 1993)

Whatever criteria we may choose to identify diagrams, there will always be representations which at least some mathematicians would take to be diagrams which our proposal will classify as formulas, and vice versa. (Stenning 2002) So, I am not going to assume that there is a fact of the matter to the question whether these and other hard cases are diagrams or not. Instead, I will focus on a particular and paradigmatic class of diagrams: geometrical ones.

Before moving on to Sherry's challenge to my hypothesis that geometrical diagrams depict geometrical objects, it might be a good idea to say a little about what other hypotheses about the nature of geometrical diagrams it excludes, and which ones it does not. For starters, it is incompatible with the claim, made by Panza (2012) and others, that geometrical diagrams are symbols, i.e., that their relation with the geometrical objects they represent is not grounded in visual similarity, but in other meachanisms of representation.[9] Furthermore, it is important to notice that depicting is a way of representing, and therefore that depictions represent. Thus, the hypothesis that diagrams are depictions is automatically incompatible with the view that diagrams do not represent. This explicitly excludes

9. As we will se a little further ahead, according to Panza (2012), the hypothesis that geometrical diagrams are depictions entails that their visual features are metaphysically dependent on those of their subject, and this is incompatible with the broadly Aristotelean thesis, defended by Panza among others, that at least some geometrical "objects inherit some properties and relations from these diagrams" (Panza 2012, 55). However, my hypothesis does not actually entail the offending claim, and so is compatible with Panza's brand of Aristotelianism, as I will argue later in this same text.

non-representationalist positions like those of David Sherry (2009), Valeria Giardino (2012) and Tarja Knuuttila (2011), according to whom diagrams are not depictions but cognitive tools. Nevertheless, my thesis is *not* incompatible with the hypothesis that, *besides* being depictions, geometrical diagrams are *also* cognitive tools. Mathematical diagrams in general, and geometrical diagrams in particular, have and still are being used with many purposes: teaching, proving, exploring concepts, etc. To say that they are depictions does not imply dismissing or neglecting their central role as tools. On the contrary, I hope the recognition of their depictive nature will eventually help us understand better their central role in these and other mathematical practices.

It is also important to remark that the hypothesis that geometrical diagrams are depictions is neutral with regards to the question whether the depicted geometrical entities are abstract or concrete. One might think that the very idea of geometrical depiction is incompatible with the view that geometrical objects are abstract and as such have no visual features that may be reproduced in depiction. If similarity of visual or spatial features underlies the kind of resemblance that mediates pictorial depiction, then it seems *prima facie* that abstract objects cannot be depicted, for they have no visual or spatial features. Even though this is an issue that deserves much more attention that the one I can devote to it here, an important remark is worth making: The hypothesis I defend here does require that geometrical entities have visual features; in particular, it requires geometrical entities to be shaped in certain ways, similar to those of geometrical diagrams. I expect circles to be round, for example, not in some *sui-generis* abstract or mathematical sense, but in the same sense that wheels and vinyl records are also round.[10] So much is true. However, it is far from clear that this is incompatible with the hypothesis that geometrical entities are abstract.

The claim that abstract objects cannot have sensible properties is not a settled matter, but instead a controversial philosophical thesis; and I am not the first philosophers to challenge it. Far from the debate on mathematical entities and their properties, a handful of philosophers have recently defended the view that some abstract objects might still have sensible features similar to those of

[10]. This does not mean that both geometrical and everyday round objects have to be both equally round. Even if geometrical circles are perfectly round in a way that wheels and CDs are not, this does not mean that both are not round, and this is all that my theory requires to say that both share at least this visual property.

everyday concrete objects (Rosen 2012, Fine 1982). It cannot be expected that I settle the issue here, yet I still want to present a couple of arguments that, even if highly controversial, might serve to make plausible the assumption that, if geometrical objects are abstract, they might not be the only abstract objects to have sensible properties.

Let us start by considering fictional characters. Several philosophers have defended and developed the thesis that fictional characters are abstract entities (Zalta 1983, Salmon 1998, Thomasson 1999, etc.). Now, ask yourselves what color the riding hood of the central character of *the little red riding hood* tale is. The common sense answer would certainly be that it is red, not some esoteric fictitious red color, but the same red as Chinese flags or ambrosian apples.[11] Colors, of course, are paradigmatic examples of sensible properties.[12] Thus, if fictional characters are abstract entities, the riding hood worn by the main character in *the little red riding hood* is an abstract object that has at least one sensible property: that of being red. From this, it follows that being abstract does not preclude an object from having sensible properties.

Consider now, to mention a second example, musical works like Shostakovich's Quartet No. 8 or Roberto Carlos' "Jesus Cristo." Common sense tells us that there is a way in which each one of them sounds. That is why we can say, for example, of a very bad performance of "Jesus Cristo" that it does not sound at all like "Jesus Cristo." When we do so, we are not comparing between two different concrete performances of "Jesus Cristo", but between a particular performance and an abstract composition. Even though they are not paradigmatic abstract objects, there are good reasons to consider musical works as abstract; after all, they are instantiated on several occasions and in different modalities, as argued by Parsons 1980 and Deutsch 1991. Thus, if we accept Parson's or Deutsch's view of musical works as abstract entities, we must accept that musical works are abstract objects that have sensible

11. As a matter of fact, we compare fictional and non-fictional objects all the time, saying things like "Sherlock Holmes is smarter than any actual detective" or "Her voice was coarse, like Miss Piggy's." This would be very hard to explain if we did not recognize that fictional entities can share some of their properties with non-fictional objects.

12. Notice that for a property to be sensible, it is enough that it be perceivable in the appropriate circumstances; it is not necessary that it be perceivable every time it is instantiated.

properties – audible properties in this case, and thus are another counterexample to the knee jerk view that abstract objects cannot have perceptual properties.

Of course, there are alternative explanations of the phenomena mentioned above (Kroon and Voltolini 2011, Sainsbury 2010, et.al.). One could say for example, that when one says that a certain musical piece sounds certain way, such talk is elliptical of saying that performances of such pieces sound, or at least ought to sound in a certain way; or one could also try to argue that, since she does not exist, the main character in *the little red riding hood* does not actually wear amy hood, red or otherwise, and that when we say that her riding hood is read we mean to say that if she existed, she would wear a red riding hood. However, these alternative explanations face major challenges that deserve significantly more attention than the one I can give them here. Also, notice that the correctness of my views does not actually depend on the soundness any of these controversial arguments. My aim here is not to settle the issue of whether abstract objects have sensible properties here, but only to challenge the claim that my hypothesis is incompatible with a very common philosophical position regarding the ontological nature of geometrical objects, i.e., that they are abstract entities. I hope to have said enough so as to allow me to assume that some abstract objects – at least geometrical figures we tend to represent in diagrams – have perceptual features. A more thorough discussion of this issue shall wait for some other occasion. Now, it is time to address Sherry's challenge.

6. Sherry's Challenge

In his 2009 article, "The Role of Diagrams in Mathematical Arguments", David Sherry explicitly offers a argument against views like mine where diagrams are pictures of mathematical objects. According to Sherry, this kind of view "is unable to explain proofs which share the same diagram in spite of drawing conclusions about different figures." (Sherry 2009, 14) According to Sherry, if geometrical diagrams were depictions, they could only be used in proof to draw conclusion about objects depicted in them or about concepts that those object fall under. Consequently, a single diagram could be used to draw conclusions about different mathematical concepts, only if it depicted objects falling under all those concepts. A diagram of an isosceles triangle, for example, could be used to draw conclusions about isosceles triangles or about triangles in general, but not about square triangles or about circles. As a corollary, if diagrams were depictions, no diagram depicting one single object could be used to draw conclusions about mutually inconsistent concepts or states of affairs. However, argues Sherry, in the history of mathematics there have been cases of mathematical proof where a single diagram is used to draw conclusions about an inconsistent set of concepts. One such famous case is Girolamo Saccheri's use of the same diagram (a bi-rectangular, isosceles quadrilateral as shown in figure 13) to prove theorems about three different and inconsistent types of quadrilaterals. According to Sherry, the existence of cases like this is enough to show that geometrical diagrams are not depictions.

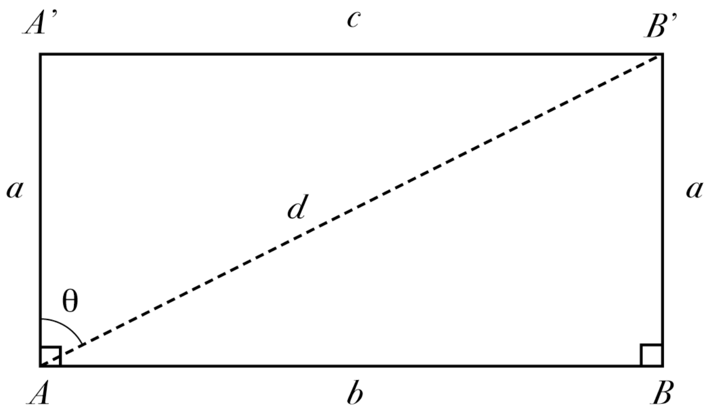

Fig 13 Saccheri's Diagram

Saccheri's overall goal in *Euclid Freed of All Blemish* (1697), was to show, as was commonly believed at the time, that Euclid's fifth postulate could be proved using the other four postulates. In particular, his aim was to geometrically construct the fifth postulate from the first four. He started his proof by presenting a quadrilateral (like figure 13) where A and B are right angles and AA' ≅ BB'. All he needed then to derive the fifth postulate was to show, using only Euclid's first four pustulates, that the aforementioned quadrilateral was a rectangle, i.e., that A' and B' were right angles too. With this goal in mind, he proved that angles A' and B' were equal and then proceeded, towards a *reductio,* to consider the possibilities that (a) A' and B' were obtuse, (b) right or (c) acute. In the end, he could not manage to derive a contradiction from possibilities (a) and (c). However, he still managed to draw mathematically interesting results from both assumptions, results that would later serve as foundations for Non-Euclidean geometry (Rozenfeld 2008).

Sherry's puzzle stems from the fact that, in going from (a) to (b) and (c), Saccheri did not switch between different sorts of diagrams, but worked with the same single figure (figure 13). When Saccheri considered the consequences of angles A' and B' being both acute, for example, he had no problem using the same diagram he had already used to draw conclusions about quadrangles with four square angles, to draw conclusions about quadrangles with two straight and two acute angles. According to Sherry, if Saccheri's diagram had been a depiction, it could have only represented what it resembled. Since it resembled a bi-rectangular quadrilateral, and not one with two right and two acute angles, it could have only depicted a bi-rectangular quadrilateral, and not one with acute or obtuse angles. However, since it was successfully used to represent the three kinds of quadrangles, it could not have depicted them, only symbolized them. Sherry's challenge, in consequence, is to explain how such a diagram could be used to draw conclusions about such different kinds of figures, while remaining a depiction. In general, the challenge is to explain how is it possible for a single diagram to depict different kinds of objects, resembling only one of them.[13] Meeting such a challenge will be the purpose of the following section.

7. A Pragmatic Theory of Geometric Depiction

My solution to Sherry's challenge is based on a puzzling, but well known fact about depiction: that resemblance is a necessary, but not sufficient, condition for depiction. In consequence, resemblance underdetermines depiction; to know how a picture looks is never sufficient to determine what it depicts. Even though there is a broad debate regarding exactly what else is necessary for a picture to depict something, there is a growing consensus in the philosophical literature that intention and context are also heavily involved in depiction (Schier 1986; Bantinaki 2008; Blumson 2009; Abell 2005, 2009; etc.). For the purposes of this article, instead of trying to characterize depiction directly, I will adopt a two-step account: First, I will give a pragmatic account of the more general phenomenon of visual representation, and then offer a characterization of depiction as a specific kind of visual representation,

13. For another historical example of this phenomenon, consider proposition III.25 of Euclid's Elements. Cf. Saito (2006: 84-90).

to know, visual representation grounded on visual resemblance.[14]

According to this account, a representational use of an object *p* visually represents an object (concept or state of affairs) *x* iff, under normal conditions, the audience of the representational act is able to work out that *p* represents *x* on the basis of (at most) what *p* looks like, the rational assumption that it was used with a representational purpose (i.e. that it was used to represent some particular object, or objects, to a certain audience), and other common background beliefs (about how the depicted objects look like, about the conventions of the media employed, etc.).[15] [16]

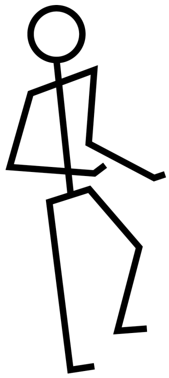

Fig 14 Stick Figure

14· I will not take a stand on the many debates regarding the relevant kind of resemblance involved, whether it is an objective relation between depiction and subject (Hyman 1999), a subjective relation between how we experience the depiction and how we experience its subject (Hopkins 1994), an audience-dependent relation between the picture-in-a-context and our memories and conceptions of how the subject looks (Newall 2006), a completely different sort of relation or a combination of these for different sorts of depictions (Freeman and Janikoun 1972).

15· In strict sense, in order to recover the representation's content, it is not actually necessary to hold these common beliefs, it is enough to accept them for the purposes of communication (Stalnaker 2002). That is why we can easily interpret pictures aimed at audiences with radically different beliefs from our own.

16· Very young children are capable of correctly interpreting very simple depictions, long before they can adopt any representational conventions or even entertain thoughts about other people's conversational intentions. Nevertheless, achieving full competence with depictions represents a major challenge in their development. (De Loache et. al. 2004; Trosethe et. al. 2004). Furthermore, cross-cultural studies (Berry 2002, Cox 1993, Derewosky 1989, Barley 1986) have repeatedly found cultural variation in our capacities for interpreting pictures. Consequently, there seems to be no cognitive or anthropological evidence for the claim that depictions are what Schier (1986) has called *naturally generating* representations: It is false that once children are initiated into a depictive system (like line drawings, color pictures, etc.) by being exposed to one picture and its content, they can easily interpret any novel picture (of an object they are independently able to visually recognize) within the same depictive system. Thus, it is very unlikely that there is something like an innate pictorial competence shared by all humans.

Notice that this broad category of visual representations covers all sorts of symbols, words, pictures and images. In order to distinguish depictions from other kinds of visual representations, it is necessary to introduce the further condition that resemblance must play a key role in determining what the picture represents. Thus, a picture *p* depicts an object *o* iff *p* visually represents *o* at least partially in virtue of *p* being made to visually resemble (or being selected because it visually resembles) object *o* in a way that is appropriate to its audience, its medium and style. This could be accomplished, for example, if the audience realizes that it would be very unlikely that the author would have given its work the visual appearance it has (resembling object *o*) unless she wanted us to recognize it as representing object *o*.[17]

On this account, a stick figure like figure 14 can be used to represent a person (to a particular audience in a given context), if it would be rational to expect from such an audience, in such a context, to figure out that, once assuming that the stick figure was used with the intention of representing something and given certain common knowledge, both about how people look, and about how monochromatic line drawings are used, that the most likely intention of the user was to represent a person.[18] Besides visually representing some man, the stick figure can also be said to *depict* a person if part of the reason why it represents a person is because it was made to look or selected by the user to look like a person.

According to this account of depiction, how a picture looks is just part of the information the interpreter exploits in order to determine what is being depicted. Consequently, what a picture depicts strongly depends on its context of use. As Calderola (2010), Dilworth (2008), Bantinaki (2008), Hyman (2006) and many others have insisted, visual resemblance is a many-to-many relation, i.e. different images may resemble the same object, and the same image can resemble many objects. As such, visual resemblance may restrict the kind of objects a picture can depict, but it cannot always

17· This is very far from being a full and satisfactory account of visual depiction. As I have already mentioned, the debate is long and complex, and I do not intend to settle it here. All I need is a sufficiently plausible account that allows me to argue for my main claim, i.e., that geometrical diagrams are depictions.

18· Notice that, so far, this account is silent as to whether a stick figure of a man actually depicts it or is a conventionalized symbol for it.

determine what it is being used to represent in every situation of use. Extra background information is still necessary. Consequently, it is not surprising that the same picture can be used to represent different things in different contexts. Consider, for example, the same single picture of a high-heeled shoe in two different contexts: one, in a shoe catalogue and another inside a crossed circle over a parquet floor (fig. 15). In the first context, the picture represents a model of shoes. In the second, in contrast, it represents high-heeled shoes in general. It is used to communicate the information that high-heeled shoes are forbidden on the parquet floor. Without being ambiguous, the picture still changes content when placed in different contexts.

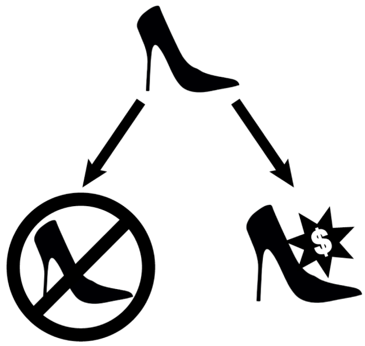

Fig 15 The Same Picture on Different Contexts

In mathematics, this kind of thing happens all the time. In different contexts, the same figure can be used to represent different entities or states of affairs (Kulpa 2009). Take figure 16, for example. The lines in the diagram look (more or less) straight, look (more or less) parallel and look (more or less) of the same size.[19] Thus, they can be used to depict and prove different things about different sorts of pairs of lines in different contexts. If the diagram is used in a mathematical context accompanied by the text "Let AB and CD be two line segments of the same length...", for example, then it is being used to depict a pair of line segments of the same length. In a different context, say one where the text accompanying the diagram said "Let AB and CD be two parallel lines ..." or "Let AB and CD be two line segments of the same length...", the very same diagram could also be successfully used to depict other things. Given than any diagram resembles many different states of affairs, we need to pay

19. For resemblance (and therefore, for depiction), the lines need not look perfectly straight, parallel, etc. After all, a more or less straight line still resembles a straight line. On the other hand however, in many cases, diagrams look perfectly round, straight, etc. even if they are objectively not (Giaquinto 2007; Newstead and Franklin 2010).

attention to the context to decide just what it depicts. This is as true of geometrical diagrams as it is for any other sort of depiction.

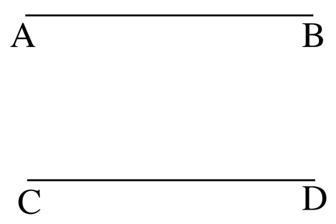

Fig 16 The Generality Problem in Geometry

One of the key claims of contemporary theories of depiction like Abell's or Blumson's is that context plays an essential role in determining the content of depictions, not only in cases of obvious ambiguity like (Figure 22), but in general when one need to determine which visual properties of the depiction correspond to similar properties in the depicted objects, and which not. Resemblance is rarely total, i.e. most pictures do not look *exactly and completely* like the objects they represent. This means that depicted objects will always have non-depicted properties (including visual ones). That is why, for example, two-dimensional pictures can be used to depict three-dimensional objects, or black and white images can be used to depict colored objects. Consider a black and white depiction of a queen of hearts. The hearts in the depicted card are red, yet the corresponding hearts in the picture are black. However, we have no problem recognizing the depicted card because of our common knowledge both of playing cards and the limitations of black and white pictures. We know that the traditional color of hearts in playing cards is red, not black. Yet, we also know that, within the limitations of black and white picturing, black is an appropriate color to use on a white background to depict a red object on a white surface. So we know that, if someone were to depict a queen of hearts in black and white, she would probably draw its hearts black.[20] Thus, even if the heart on the picture resembles a black heart more than it resembles a red heart, we conclude that it makes more sense to infer that it depicts a red heart that it does a black heart.[21] Context, therefore, help us determine which visual properties of the picture

20· This does not mean that the same picture, in a different context could not depict a different object. If the hearts of playing cards were blue instead of red, for example, the same picture could also be successfully used to represent a queen of blue hearts.

21· Consequently, determining the content of a depiction is not a matter of determining what it resembles the most.

are to be taken as standing for analogous properties in the depicted object, and which not.

The moral of the theory is that, when dealing with the depiction of objects, one must be very careful to distinguish between depicting an object that is *F* (for some visual feature *F*) and depicting an objects **as** being *F*. Black and white pictures, for example, can depict colored objects, but they cannot depict them *as* colored. The hearts in the aforementioned card are red, yet they are not depicted as being red. They are not depicted as being black either. The black and white picture is *silent* regarding the color of the depicted heart.[22] In general, most if not all depictions are silent regarding one or another feature of their subjects. The picture on my voter ID, for example, is silent about my weight and about whether or not I was wearing pants at the time my picture was taken. This is just a normal feature of depictions. A picture does not have to be *F*, or even look as being *F*, to be used to depict an object that is *F*. This is what allows objects that are *not F*, and do not even look like being *F*, be used to depict entities that *are F*; this is why a black and white pictures can be used to depict colored objects, two-dimensional pictures can be used to depict three-dimensional objects and figures in Euclidean space can be used to depict geometrical figures in non-Euclidean space.[23]

One must be very careful in noticing that to say that a representation looks like or visually resembles its subject does not mean that looking at the representation is *just like* looking at its subject. All it means is that there are some properties that the depiction shares with its subject that are also *visible in* the depiction. This means that for the desired resemblance relation to hold it is not necessary that the same property that is visible in the depiction is visible also in the subject. When one thinks of everyday cases of depiction, i.e., photographs, realistic paintings, etc., the properties of the subject reproduced in the depiction are properties that one can also see in the subject. However, not all depictions are like that. As a matter of fact, many of the depictions used in science are not like that. In many of them, what we see in the depictions could not be seen directly on the depicted things themselves. Micrographic, telegraphic and stroboscopic pictures, for example, let us see things that are not visible to the bare eye. Consider the aforementioned example of photo finish photography: there is

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22. Schier talks of "representational commitments" to make this very same point. In his words, our use of the black and white picture holds no commitment regarding the color of what it depicts.

23. I thank Alex Grzankowski for raising this issue.

a sense in which what we see is similar to how the last instant of the race would look like, but of course we cannot actually see such instants (Canales 2009).

In consequence, when I claim that a diagram D resembles a geometrical object O all I claim is that there is at least one property P such that (i) both D and O are P and (ii) one can perceive that D is P. Consequently, when I say that D depicts O in a proof, all I mean is that D represents O at least partially in virtue of being drawn so as to share this property P in a way that under normal conditions, the readers of the proof can be rationally expected to be able to use this information to work out its content – i.e., that it represents O.

As argued in the first two sections of the article, when aiming to represent a geometrical state of affairs in a proof, one must decide both what must be represented and how. To determine what must be represented, one must consider what is given in the initial conditions of the proof, what has already been proved, etc. To determine how should one represent it, one has to evaluate the logical and cognitive advantages and disadvantages of the different means of representation available. In particular, one must decide what information is worth depicting in the diagram, and what information is better left in the text. To determine whether a feature of the target geometrical situation is worth reproducing in the diagram, one must weigh its costs and benefits. In Euclid's example above, for example, we use a roundish closed curve to represent a circle. it is just roundish, because drawing it perfectly round would require too much effort without adding much new relevant information. On the other, if the line was not closed, but open, it would lacked a feature of circles key to the validity of the proof (Panza 2012) and if it was not roundish but polygonal, the resulting diagram would have been too confusing. Thus, we conclude that being a roundish closed curve is a feature of the circle that is worth reproducing in its depiction, while perfect roundness is not.

In a similar fashion, in Saccheri's case we use a quadrangle with four more-or-less square angles, to represent a quadrangle with two straight and two acute angles, because trying to make the depiction even more similar to its target geometrical object would have be too difficult, within the limitations of our medium. Still, we try to include in the depiction as many features given in the setting of the problem as possible. The setting of the problems asks us to consider a (i) quadrangle ABCD such that

(ii) A and B are square, (iii) A' and B' are acute, and (iv) AA' ≅ BB'. It is quite easy to draw a quadrangle that more or less satisfies three of the previous constraints, but not one that satisfies all four, so we have to decide which constraints the diagram will satisfy and which one not. Saccheri chooses not to include (iv) in his diagram, however we could have chosen to exclude a different constraint, and work with a different diagram. For example, some contemporary textbooks use a diagram like figure 17, that satisfies (ii), (iii) and (iv), but not (i).

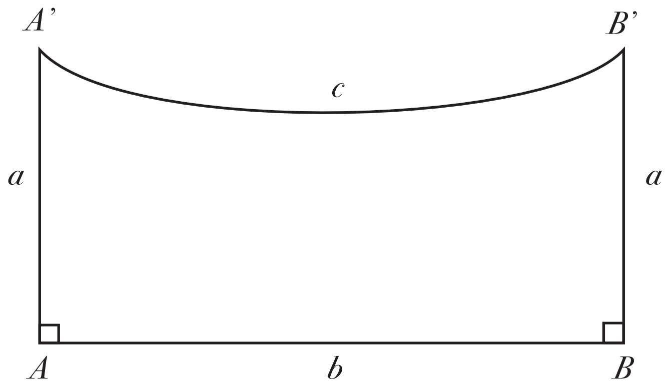
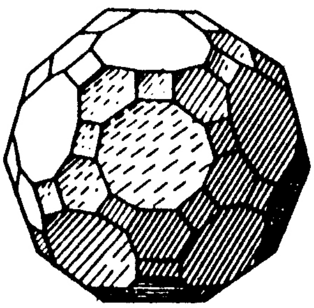


Fig 17. Alternative Saccheri's Quadrangle

Whatever constraint we do not include in our diagram, still has to be communicated in considered in the proof. This is commonly done textually, but there are other mechanisms we can also use. For example, a little quadrangle is usually added on the angles that are to be interpreted as straight (See figures 16 and 17). These symbols are not part of the depiction itself, but auxiliary symbols that are added on top, just as the letters used to identify the angles.



7.

Fig 18 Two-Dimensional Depiction of a Three-Dimensional Object (Kepler 1619/1997)

In the end, what features we decide to include in the depiction will depend on the costs and benefits of including or excluding them. When we cannot include in the digram all the information given in the setting of the problem, we have to choose which information to exclude and make sure there are enough indications, either in the accompanying text or symbols, as to what information is missing from the diagram. This is why the resemblance involved in depiction is rarely total: Most of the times, it is not worth reproducing all the properties of the depicted object in the depiction; it might even be disadvantageous. Accordingly, most depictions have properties (including perceptual ones) that their depicted subjects do not have, and vice versa. This is why, for example, two-dimensional pictures can be used to depict three-dimensional objects (as in figures 18 and 19), black and white images can be used to depict colored objects and diagrams in Euclidean space can be used to depict figures in Non-Euclidean space. Just as a black picture can depict a colored object, geometrical figures of one sort can be successfully used to depict geometrical objects of a geometrically different kind. Sometimes, the diagram of a triangle *is just* a triangle, but this need not be so. Just as a picture of a circle need not be a circle itself, a diagram of a cube need not be a cube. Diagrams are two-dimensional entities, yet they are also used to represent mathematical entities in three or more dimensions. Consider figures 18 and 19. The two-dimensional figure represents a three-dimensional object without being a three-dimensional object itself. It is enough that it resembles it in the relevant way, that is, the way that helps the depictions' intended audience recognize that it is indeed three-dimensional. The (concrete, bi-dimensional) lines in the diagram represent (abstract) tri-dimensional lines without being (concrete) tri-dimensional lines themselves. It is enough that they resemble them in the relevant way.

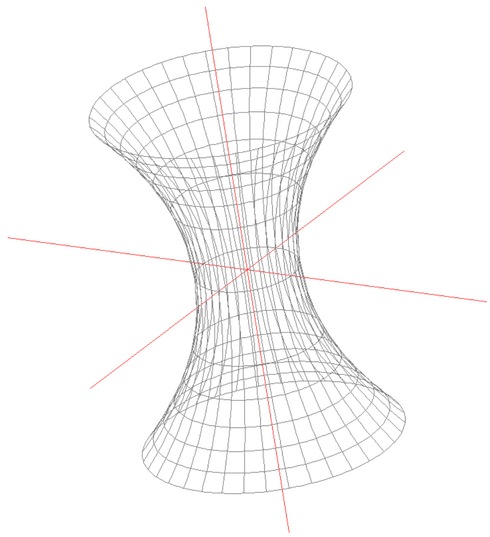


Fig 19. Bidimensional Representation of a Tridimensional Object

The same thing happens in the diagrammatic representation of objects of non-euclidean geometry. The picture in figure 20, for example, is an object in Euclidean space, but it is here used to represent Morley's triangle in hyperbolic space.

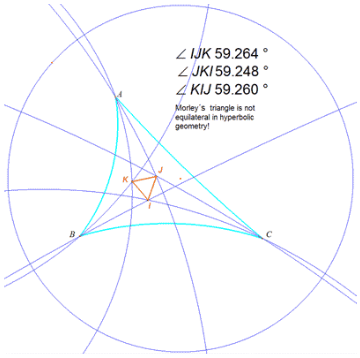


Fig 20. Euclidean Representation of a Non-Euclidean Object

The same thing happens in Saccheri's case. Even though the diagram is a figure in Euclidean space, it can depict quadrangles in both Euclidean and Non-Euclidean space. In general, for any property *F*, objects that are *not F*, and do not even look like being *F*, can be successfully used to depict entities that *are F* as long as there are other similarities and contextual clues that allow the interpreter to identify the depiction's content. How context helps the interpreter of a diagram fix its referent will be explained in detail in the following section.

8. Content and Context in Geometrical Depiction

To further detail how context helps fix the content of depictions, it might prove helpful to say a little bit more about how context is exploited in human communication. For the purposes of this paper, let me adopt the well known account owed to H. Paul Grice (1975), according to whom, whenever we engage in conversations, our communication is guided by a set of assumptions or maxims. These maxims include: (maxim of quality) say only what you believe to be true and of which you have enough adequate evidence; (maxim of quantity) be as informative as necessary, (maxim of relation) contribute only relevant information to the conversation; (maxim of manner) and be clear. These maxims together conform what is known as the cooperative principle: "Make your conversational contribution such as required, at the stage at which it occurs, by the accepted purpose of the talk exchange in which you re engaged" (Grice 1975, 46). Appealing to this maxim has proved to be helpful in explaining how we exploit contextual information to resolve ambiguities, fix extension to predicates, understand sarcasm, etc. Assume now our use of pictures follows Grice's cooperative principle.[24] In particular, assume Saccheri's use of figure 13 to illustrate the hypothesis that A' and B' are obtuse angles adheres to this principle. Now, the four angles of of Saccheri's parallelogram are right. Angles A' and B' clearly look right. They resemble right angles, but also resemble (among other things, and to a lesser degree) other sorts of angles and therefore could be used to depict them. Thus, it is necessary to consider which of these possible interpretations is most likely to be the one intended by the author, Saccheri. Without further information, the most promising hypothesis is that the diagram depicts a figure very much like itself, i.e., a parallel quadrangle. However, after reading the accompanying text, we realize that interpreting Saccheri's diagram as depicting such a parallelogram would violate Grice's maxim of quality, since it would be inconsistent with the information contained in the text. Thus, we infer that A' and B' must actually depict obtuse angles, not right ones. This restores consistency to Saccheri's combination of picture and text, and allows us to maintain the presumption that he was adhering to the aforementioned principle of conversation in using the diagram in that context.

Sometimes, in order to restore consistency between text and diagram, one must reject not what the diagram shows, but what the text says instead. Consider for example, figure 21 as used in Euclid's reductio proof in I.6 (See Netz 1999, p. 55).

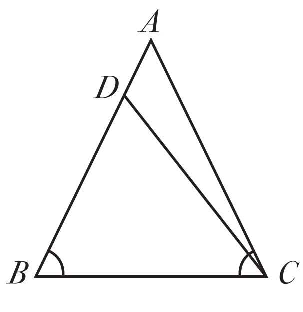


Fig 21 Euclid I.6

24. I am not the first one to suggest this, of course. Grice himself thought the principle applied to all rational, cooperative practices, not only verbal conversation; while Fling Schier (1986) and Catherine Abell (2005) have already applied Grice's principles to depiction.

In the diagram we see two triangles sharing one side (BC) and one angle (DBC), as sated in the initial conditions of the proof. We also see that one of the triangles (BCD) is inside the other (ABC) and, consequently, is smaller. Finally, we also see that angles ABC and ACB are more or less equal. The accompanying text confirms that the angles they represent are equal. It also asks us to work under the hypothesis that DB = AC. Thus, we assume that the depicted triangles ABC and BCD are of different sizes, have two equal sides (BC = BC and DB = AC) and one equal angle (ABC = DBC). However, we know from previously proved results, that if two triangles have two equal sides and one equal angle, one cannot be larger than the other, which contradicts what we see in the digram. As in Saccheri's case above, we have reached a contradiction. Since we need to restore consistency in order to determine the diagram's reference, and the contradiction is easily avoided if we reject the hypothesis under consideration, we do that. Once we stop trying to interpreting lines DB and AC as equal in length, we can easily identify the depicted figures as two triangles ABC and BCD such that ABC > BCD, ABC = DBC, BC = BC and AB > DB. These, of course, are not impossible triangles, but regular possible triangles. This way, we can make sense of what happens in reductio proofs without having to postulate impossible geometrical objects. In general, in reductio proofs of this sort, the diagram does not represent the hypothesis to be reduced (or the contradiction reached from it), but the positive conclusion we obtain from the reductio. If a reductio proof assumes that not-P to get to a contradiction and thus show that P, we can expect its diagram to depict a situation where P holds, not one where the reduced hypothesis not-P holds, for this is impossible.

Identifying the content of Saccheri's parallelogram requires significant input from the context. Angles *A* and *B* in Saccheri's diagram do not look obtuse or acute for sure; but an object does not have to be or look as being *F* in order to depict an object that *is F.* A figure that is not *F* can depict an object *a* that is *F*, as long as it resembles *a* in some other respect. Since visual resemblance need not be total to ground depiction, an object can depict another without sharing all of its visual features. Absolute similarity is not necessary. In particular, a straight angle can depict an acute angle, as long as they share enough other visual properties for the audience to be able to identify one as representing the other. In the case of Saccheri's parallelogram, we have enough visual clues in the diagram's geometrical features to identify each and every angle. We can identify angle *AB* in the diagram, for example, just by identifying the meeting of lines A and B, disregarding whether such angle is straight, acute or obtuse. In this regards, Saccheri's parallelogram is not very dissimilar from the black and white picture of the queen of hearts above. Even though the heart in the black and white picture *is* black, the heart it depicts *is not* black. Even though the angle in the diagram *is* straight, the angle it depicts *is not* straight. In both cases, how the picture looks seriously underdetermines what it depicts, and we need to appeal to our substantial common knowledge about the world to determine what is being depicted. In the case of the queen of hearts, we had no problem interpreting the black hearts in the picture as red because of our common knowledge both of playing cards and the limitations of black and white pictures. Similarly, in Saccheri's case, we have no problem using both our knowledge of geometry and of the conventions and limitations of (two-dimensional, black and white) geometrical diagrams (in Euclidean space) to interpret the diagram.

Disregarding the underdetermination of depiction by resemblance makes it hard to understand how the same diagram (or different tokens of the same diagram type) could be used to depict different mathematical objects. Saccheri's parallelogram is mysterious only under the wrong impression that it is sufficient to look at a diagram to get to its content (Larkin and Simon 1995, apud. De Giardino 2012). Yet, once we recognize the importance of context in determining what is depicted, we realize that there is nothing mysterious in Saccheri's parallelogram. Without a proper understanding of the role of context in depiction, Sherry cannot see how the diagram can change its content from one quadrangular to another. So, he takes the radical anti-realist and anti-representationalist alternative of claiming that mathematical diagrams (and mathematical formulae) do not represent mathematical objects at all. However, once we understand that how a diagram looks underdetermines what it depicts, no such radical move is required. Drawing only one parallelogram, all three parallelograms can be depicted. Thus, Sherry's challenge poses no real threat to the thesis that geometrical diagrams are depictions.[25]

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25. Notice that the account given to Saccheri's parallelogram is readily available for the many other cases where geometrical diagrams lack some of the relevant properties of the geometrical figures they depict, for example, the diagrams used to represent impossible situations in some of Euclid's *reductio* proofs, or the diagrams representing polygons inscribed in circles with polygon sides drawn curving inward in the Archimedes Palimpsest. In all these cases, the substantial differences between diagram and figure pose no challenge to the depictive hypothesis.

9. Creative ambiguity and *seeing-as*

A challenge to the hypothesis that diagrams are depictions similar to Sherry's has been raised by Danielle Macbeth on her otherwise excellent study of Euclidean diagrams (Macbeth 2009, 2010 and 2014). According to Macbeth, that the role of diagrams in mathematical proof largely consists in the de- and reconfiguration of content displayed by geometrical drawings, not in the analysis of a given static picture. Consider, for example, the proof of Euclid I,1 presented at the beginning of this text. Notice how it is essential to be able to regard one and the same drawn line AB now as the radius of a circle and then as one side of a triangle. According to Macbeth, this is incompatible with the thesis that geometrical diagrams are depictions, since the content of a depiction cannot change in the course of reasoning about what it depicts (Macbeth 2009, 252). Thus, we need an account of the content of Euclidean diagrams that allows for shifts in content and for what she calls, following Manders (1996, 2008), the *popping up* of new geometrical information from the diagram.

> In a Euclidean demonstration, what is at first taken to be, say, a radius of a circle is later in the demonstration seen as a side of a triangle. But how could an icon of one thing become an icon of another? How, for example, could an icon of a radius of a circle turn into an icon of a side of a triangle? (MacBeth 2009, 252)

To account for the shifting content of Euclidean diagrams, Macbeth endorses a pragmatic account very similar to mine, where the author's intensions, as manifest in the diagram's accompanying text, play an essential role in determining its content. According to her,

> "…the Euclidean diagram can mean or signify some particular sort of geometrical entity only in virtue of someone's intending that it do so and intending that that intention be recognized. One's intention in making the drawing—an intention that can be seen to be expressed in the setting out (in those cases in which there is one) and throughout the course of the kataskeue—is, in that case, indispensable to the diagram's playing the role it is to play in a Euclidean demonstration." (Macbeth 2014, 82)

Furthermore, one's intention can override what the diagram shows, so that if the geometer draws an angle with the intention "merely to draw an angle,… that which he draws… will necessarily be right, or acute, or obtuse; but [what it represents] will be neither right nor acute nor obtuse. It will simply be an angle." (Macbeth 2014, 82)

According to Macbeth, the recognition that intensions play an essential role in determining the content and role of a diagram entails that they cannot be depictions, i.e., that a figure drawn in a Euclidean diagram, even though it may also resemble its object in appearance, it does not represents it in virtue of this resemblance in appearance. (Macbeth 2014, 95) Macbeth recognizes that diagrams of circles, for example, look like circles, but argues that it is not because of this that they represent circles, but because of the interplay between pragmatic mechanisms and a structural homomorphism between the parts of the diagram and those of the geometrical objects it represents.

Thus, for Macbeth, a pragmatic account that takes seriously the importance of intensions is incompatible with the hypothesis that diagrams are depictions (Macbeth 2009, 252-3). I hope to have shown here that this is just not so, i.e., that Macbeth is wrong in thinking that the content of depictions is fixed previously and independently of any pragmatic considerations regarding the intensions of the mathematician. Thus, in the end, Macbeth and me are very close in agreement regarding the use and interpretation of Euclidean diagrams. However, we disagree on whether this means that they are depictions or not.

Let me take some time to develop the thesis that the kind of shift in *seeing a diagram* that is required for proofs like Euclid I.1. can be accounted within a framework like mine, i.e., that recognizing that the use of diagrams in Euclid requires actually *seeing* the same diagram (or parts of a diagram) as different things in different moments, is compatible with my claim that diagram are depictions, for depictions can also be seen in different ways, i.e., sometimes to adequately interpret an image, it is necessary to recognize a shift in what it depicts.[26] In other words, what Grossholz has called "creative ambiguity" is a feature common to depictions, thus it is far from being incompatible with it. Even taking into consideration the conventions and techniques at play, a single image can equally resemble more than one object and ambiguity may result, and this ambiguity can be

[26]. Of course, the most famous image, the paradigmatic example of shifting pictorial content in the literature, as Macbeth herself recognizes, is Wittgenstein's famous duck-rabbit. However, according to Macbeth, for the very same reasons that she claims that the diagram in Euclid I.1. is not a depiction, is not a depiction either.

exploited in the conveying of a message. Consider the following example: In 2004, the Light of Life foundation ran a series of ads with the purpose of raising awareness of the growing number of cases of neck cancer among women. The ad featured a cropped photograph resembling a woman's naked torso (figure 22), showing part of her waist, one of her breasts and just the edge of her nipple. However, the image was actually a cropped photograph of a woman's face, showing just her lower lip, chin and long neck. Even if we take in consideration the usual conventions associated with color photographs, without further input from its context, it is impossible to determine whether it depicts a fragment of woman's naked torso (part of her waist, one of her breasts and just the edge of her nipple), or a fragment of a woman's face (a tip of her lower lip, a quarter of her chin and half her long neck). It resembles both things equally. The image also contained the legend "The fastest growing cancer among women is not what you think". The photograph, by Frank W. Ockenfels, was purposely ambiguous between both interpretations – torso and neck – and it was this ambiguity that made it a perfect fit for harnessing the message that the foundation wanted to communicate: that the fastest growing cancer among women is not what most people think, i.e., breast cancer, but neck cancer. [27] In order to successfully interpret the ad, it is necessary to interpret the photograph as depicting once a torso and then a neck. This might require seeing, for example the same crimson part both as a nipple and then as a lip, but this does not make the picture no longer a depiction.

Thus, the possibility of creative ambiguity is not only consistent, but a consequence of the fact that isolated from their context, pictures do not depict anything, but only when placed in an adequate context. In other words, substantial input from the context is necessary to determine what the image represents. If the context changes, what the picture depicts might change as well. If the context is dynamic, as it is in geometrical proofs[28]

27. The example is described in Meyers 2004.

28. Notice, however, that the discarded interpretation still plays some role in the interpretation of the picture. It was precisely this depictive ambiguity that was exploited by the *Light of Life Foundation* to raise awareness of thyroid cancer. Similar representational ambiguities have been productively exploited in mathematics (Grosholz 2007; Giardino unpublished).

Fig 22 Ambiguity in Depiction

Given the importance of contextual information in determining the content of depictions, that the same depiction can change content from one context to the next must not be surprising at all. Thus, when we have a dynamic context like in the example above, it is not surprising that the same depiction switches from depicting a torso to depicting a neck, from depicting a nipple to depicting a lip. Similarly, when the proof so requires, we might be able to interpret the same line as depicting a side of a triangle one time, and as depicting a radius of a circle.

This phenomenon has been tried to be explained in terms of *seeing as*, i.e., that the reason why the diagram is useful in proof is because of the insight we get from first seeing the same drawn line as the radius of a circle and then seeing it as one side of a triangle. In other words, some philosophers have tried to assimilate what happens in proofs like Euclid I,1 with Wittgenstein's famous drawing that can be seen both as a rabbit and as a duck. However, both cases are radically different. In seeing Wittgenstein's drawing as a picture of a duck, one must assign an interpretation to the drawing such that certain part of it represents its beak, another its eye, etc. If we want to see it then as a picture of a rabbit, we must see the part that represented the duck's beak as now representing the rabbit's ears, and so on. Abandoning one interpretation is necessary, before adopting the new one, because both interpretations are inconsistent. Nothing can be both a rabbit ear and part of a duck's beak. However, nothing of the sort is necessary in interpreting the diagram associated with Euclid I,1, for there is nothing inconsistent in a line being both the side of a triangle and the radius of a circle. After all, both the property of "being the side of a triangle" and the property of "being the radius of a circle" are extrinsic, relational properties, i.e., properties that a line has not because of any of its inherent features but because how it is related to other geometrical objects. As Emily Grosholz has emphasized, "Because the side of a triangle is a line, it is intelligible independent of the triangle, despite the fact that regarded as a side it is intelligible only in relation to the triangle as a whole." (Grosholz 2007, 36) Nevertheless, being related in one way to a geometrical object does not preclude the same line from being related in different ways to other geometrical objects. Thus, in explaining how we are able to regard line AB both as the radius of a circle and as one side of a triangle, there is no need to appeal to "seeing as". In order to make sense to the double role the line plays in the proof, its is enough to notice that when we describe AB as a radius, we pay attention to some of its extrinsic features, while we focus on different extrinsic features when we describe it as part of a triangle. In Macbeth's words (2014), "it is the shift in one's perceptual focus that effects what we would otherwise think of as a step in reasoning." When we regard it as a radius, we focus on its relation to the circle that has A as its center; when we regard it as the side of a triangle, we focus on its relation to lines BC and AC. There is nothing here that could make us abandon the thesis that diagrams are depictions.

10. Panza's Challenge

At least since Plato, it is received wisdom that depictions are metaphysical derivative from their subjects (Plato 1892) or, to put in layman's terms, depictions look the way they do because of how what they depict looks like, and not the other way around. The basic idea behind this widespread intuition is that to depict is to somehow reproduce some of the visual properties of the subject. Consider an everyday realist painting of a sunset by the sea. The painter paints the sky tones of red and orange, because that is how the sky looks at sunset. It paints parts of the sea a greenish blue, because that is the color of the sea at sunset. In the end, if the painter succeeds in depicting the seaside landscape, her painting will look the way it does, among other things, because of the way the landscape itself looks. In general, the painting has the relevant visual properties it has – the ones that it shares with its subject – because of the way the landscape looks, that is, because the landscape had those visual properties first.

In the case of geometrical diagrams, this means that, if they were depictions, as I argue, then their visual properties should be grounded in the visual properties of the geometrical objects they depict. Geometrical diagrams of circles would be round (or, at least, roundish), for example, because the circles they depict are round themselves, and not the other way around. However, argues Panza (2012), we have strong historical evidence that the objects of Euclidean geometry actually inherited some of their properties – including some of their visual and spatial properties – from the diagrams used to study them, and not the other way around. Consequently, we must reject the hypothesis that diagrams are depictions.

Panza's challenge against the hypothesis that geometrical diagrams are depictions is based on a broadly Aristotelian interpretation of Euclid's use of diagrams in his *Elements*. As I have mentioned at the beginning of this chapter, proposition I.1 of Euclid's Elements requires to construct an equilateral triangle on a given line segment AB (fig. 1). To achieve this, Euclid describes two circles with centre in the two extremities A and B of the given segment, and takes for granted that these circles intersect each other in a point C. However, it has been known for a long time that this is not licensed by his postulates. This has led to a longstanding debate on whether Euclid's proof is flawed, or it is warranted on other grounds. In his (2012), Marco Panza argues that Euclid's argument is sound, but diagram-based, meaning that its soundness depends on the existence of what he calls "diagrammatic attributes", i.e., attributes that geometrical objects inherit from the diagrams that represent them (through a process of abstraction in the Aristotelian sense, which Panza takes to consist, broadly speaking, in isolating shared properties of objects and them treating them as if they were objects). In other words, Panza argues that the best way to understand proofs like this is by adopting a broadly Aristotelian framework, where abstract geometrical objects are not metaphysically prior to the concrete diagrams we use to study them, but instead can inherit some of their properties from them. On this view, geometrical circles are round, for example, because the diagrams that depict them are paradigmatically round, and not the other way around.

I do not want to address Panza's interpretation of Euclid here, for I cannot pretend to have the historical credentials to challenge his take on this piece of ancient mathematics. Instead, I plan to meet his challenge in a different way. I want to argue that my thesis that the diagrams of Euclidean geometry are depictions is compatible both with Platonism - the claim that geometrical objects are metaphysically prior to geometrical diagrams – and Panza's Aristotelian claim that geometrical diagrams, despite being concrete, are more fundamental than geometrical objects, at least with regards to some geometrical attributes. I will do this by arguing against the widespread intuition that depictions look the way they do because of how what their subject matter looks like, and not the other way around. I will argue that, once we understand exactly in what sense depiction is grounded in similarity, we will be able to see that the thesis that diagrams are depictions is compatible with both Platonism and Aristotelianism.

11. Depiction and Similarity

My defense against Panza's challenge involves a double strategy: one negative and one positive. On the negative side, I will attack the admittedly intuitive thesis that depictions look the way they do because of how their subjects looks like, and not the other way around. Yet, in the positive one I will try to recover what is right about our intuition that objects are somehow metaphysically prior to their depictions. And then I will show that once we properly understand the sort of metaphysical priority involved, it will be clear that it is compatible both with Aristotelianism and Platonism. I will proceed thus in order, from the negative to the positive.

The main point behind my negative thesis is that visual similarity can ground depiction even if it is not created ex-profeso to ground depiction. Let me start by noticing that it is true that, in many paradigmatic cases of depiction, like a painter depicting a landscape as it unfolds before her eyes or a sculptor chiseling marble to depict a heroic figure, the resulting depictions look the way they do because the artists *shaped them* with the explicit intention of reproducing certain visual properties of preexisting objects or situations, i.e., so that the final results visually resembles them. Hence, it makes sense to say that the resulting painting or sculpture has the shape or the colors it does *because of* the shape and colors of the object or scene it depicts. However, this is not something that can be generalized to any kind of depiction. Consider the existence of pictorial ready-mades and found object sculptures, i.e., cases where already-existing objects (including pictures) are used to create new pictures or sculptures that depict objects that were not already depicted by the original objects. In these cases, it would be a mistake to say that the resulting pictures owe their shapes and looks to the objects they aim to depict.

Consider, for example, that when David Kemp took a bunch of boots and assembled them to depict a group of dogs in his public piece *Thinners Dogs* (2010) (Figure 23), he did not give the shoes the appropriate shape necessary to depict dogs. Instead, he exploited the preexisting similarity between the boots and the different parts of dogs in order to create his sculptures. In the end, his assembled sculptures visually resembled dogs, and in virtue of this resemblance they succeeded in depicting them. However, it would be a mistake to say that the boots Kemp used to create his sculptures had the shape they do because dogs had the shape they do. No, those boots had already their shape independently of the shape of dogs. Similarly, cucumbers had the shape they did before Sarah Lucas appropriated them to depict male genitalia in her *Au Naturel* sculpture from 1994 (Figure 24). In these and many similar cases, the relation of visual similarity preceded the relation of depiction, both on a temporal and metaphysical sense.

Fig 23. Thinners Dogs, David Kemp (2010)

In general, in cases where the artist does not shape, paint or in general makes objects look the way they do, but instead chooses and assembles preexisting objects because of the way they *already* look, it would be a mistake to say that those objects the artist picked looked the way they did because of how the object the artist wanted to represent looked. This means that visual similarity can ground depiction even if it is not created *ex professo* to ground depiction. In general, what depiction requires is visual similarity, but it is completely neutral to whether visual similarity is the result of the process of depicting or is previous to it. In other words, my hypothesis is neutral regarding the metaphysical priority of the shared visual properties.

Fig 24. Au Naturel, Sarah Lucas (1994)

Yet one might be tempted to say that, even if some of their parts do not, the aforementioned examples *as a whole* still look the way they do because of how their subjects look and that therefore they are not counterexamples to the general thesis. And that might be true, but it misses the point of the criticism, for we can at least imagine cases where the artist picks a single object or image to depict her subject without in any way modifying its appearance or adding any other material component; as long as the chosen objects bears enough visual similarities with its subject, the case is not implausible. In those cases, we need to say that the object succeeds in depicting it subject even though it does not look the way it does because of the way its subject looks. This is enough to show that the main argument behind Panza's challenge is unsound, for one of its premises is false.

Still, there is something to Plato and Panza's intuition that the depiction is somehow metaphysically derivative of its subject. You might still think that, even in the case of collages and found objects, the objects that make up the depiction may look the way they do independently of what they are being used to depict, yet the depiction does not, for there is still a substantial sense in which the chosen object or objects would not constitute a depiction of their desired subjects unless they looked like them, and that this is what establishes the metaphysical priority of subject over depiction. If the object did not look the way it did it would not have been chosen by the artist and this choosing and assembling is metaphysically analogous to chiseling stone or painting on canvas, i.e., it is guided by the artist's aim of finding an object that reproduces some of its subject's visual features. This much is true, and I think there is something right about thinking that an object would not be a depiction of something unless it looked like it. Yet, the best way to cash out this intuition is not to postulate some metaphysical priority between depiction and subject, as I have shown above, but between depiction as a *relation* and the way subject and depiction look, in particular, the fact that they are visually similar. However, to do so would be nothing but to affirm the basic fact that depiction is partially grounded in similarity. This means that an object x depicts a subject y partly in virtue of sharing some of its visual properties, i.e., the fact that x depicts y is partially grounded on the fact that x is visually similar to y. The fact that x is visually similar to y is in turn grounded on x and y having the visual properties they do. Thus, visual similarity is metaphysically prior to depiction, and I think this is all there is to the putative priority of how the subject looks over how the depiction looks.

Now, notice that the priority of similarity over depiction is in no way violated by Aristotelianism. It might be true that, as Panza argues, Euclidean objects inherit some of their properties from the diagrams used to study them, and thus that the visual properties of Euclidean objects are metaphysically grounded on the analogous properties of Euclidean diagrams, but this in no way violates the principle that their sharing of this properties is metaphysically prior to the fact that one depicts the other. In other words, the claim that depiction is grounded on visual similarity is compatible with both Platonism and Aristotelianism, for it does not matter whether how geometrical diagrams look is metaphysically prior to how geometrical objects look, as long as they look similar enough for the former to depict the later.

12. Open Questions

In this paper I have tried to offer an account of diagrams grounded in two main principles about our general use of representations. The first one is that representations used to make inferences are shaped by both logical and cognitive constraints, and diagrams are not an exception, The second one is that diagrams are depictions and as such they exploit perceptual resemblance to fix their reference. I have tried to show how combining these two insights can throw some new light on some of their otherwise puzzling questions, like why the same diagram can be used in different contexts to represent different things, how do text and diagram interact in proof, and what does the diagram in a reductio proof represents. I have tried to show that as a consequence of the logical and cognitive constraints that shape diagrams, their visual resemblance to what they depict is usually not complete but partial, and this results in an underdetermination of their reference. In other words, I have tried to show why, in diagrams, just as in depiction in general, visual resemblance constraints but does not fully determine reference. Most times, the diagram will resemble more than one different state of affairs, and we will need extra information to identify the intended referent among them. Thus, it is necessary to combine the information we perceive in the diagram with the information from the accompanying text to determine the content of a diagram. This allows for a more dynamic and malleable use and interpretation of diagrams, as is manifested in proofs like Saccheri's, where the same diagram is used to represent mutually inconsistent geometrical objects, on Euclid's I.6 where the diagram is used to reduce a hypothesis to contradiction.

I hope to have shown how taking diagrams to be depictions help us understand important aspects of their role in geometrical proof. However, even if I am right about the pictorial character of geometrical diagrams, a few questions remain open. For starters, we still have the issue of whether other mathematical objects have visual properties (Maddy 1990, Lomas 2002, Levine 2005) and thus whether there could be pictures in other mathematical fields, remains open as well. One must look at the different kinds of diagrams employed in mathematics case by case. In some of them, like the projective diagrams of knot theory or some diagrams in topology (such as the ubiquitous torus), the answer will probably be yes, they are depictions (Brown 2008). Yet, it very unlikely that the same can be said of all cases. Logical diagrams like those of Peirce, Venn or Euler, for example, are most likely not depictions. A shaded intersection of circles does not look at all like the intersection of sets, if such a thing even looks like anything. In a similar fashion, Penrose's graphical tensor notation (1971) (figure 25) is not pictorial either. Notice that one of its conventions is to use a straight line to represent symmetrisation and a wiggly line to represent anti-symmetrisation; yet, there is nothing straight in symmetrisation that is wiggly in antisymmetrization. Consequently, those lines do not depict such operations, only symbolize them. In general, it should not be much of a surprise to notice that as branches of mathematics become more abstract, their diagrams become less pictorial and more symbolic. Thus, much of the most prevalent kind of diagrams used in mathematical today, like the diagrammatic notation of quantum group theory, Cvitanovic's birdtracks, string diagrams for monoidal categories, planar algebras, etc. are not pictorial, but symbolic. Even in more basic areas of mathematics like simple arithmetics, diagrams such as those used in so-called visual proofs (Nelsen 1993) (figure 12) do not seem to be depictions either (Brown 2008).[29] So, in the end, I hope one of the lessons to learn from the hypothesis here developed is that mathematical diagrams do not form a single natural kind, and thus their representational nature must be studied carefully depending on the kind of diagrams one is interested in.

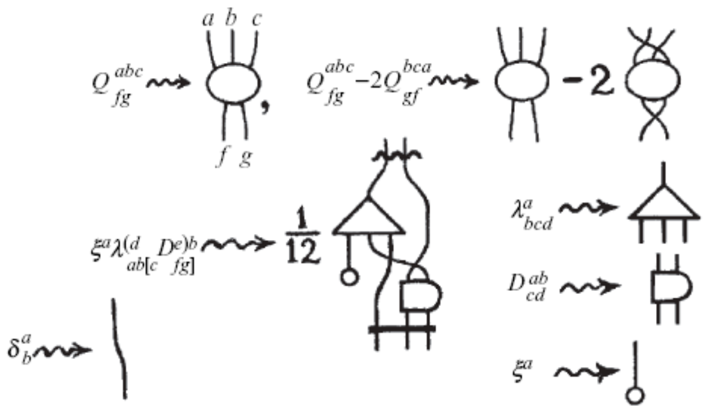

Fig 25. Penrose's graphical tensor notation

Another set of important issues surround the very notion of depiction and the symbol/picture distinction. How deep (or shallow) is it? Is it a distinction that exists only at the level of pre-theoretical intuition, or does it have further significance, perhaps at the cognitive level? The empirical data suggests that different cognitive capacities underlie our ability to interpret pictures and symbols (Farah 1989, DeLoache and Burns 1994, DeLoache 1998, Bloom and Markson 1998, Bovet and Vauclair 2000, Uttal et. al. 2006, etc.). However, the very distinction is elusive. Some empirical studies on the cognitive aspects of pictures contrast it only with linguistic representations, and not with other sorts of visual representations, so there is little empirical basis for an answer. The same can be said about the work done on the development of visual representations (Morley &

29·. According to Brown (2008), even though they are pictures, these diagrams do not depict what they represent. Their content is infinite, and therefore cannot be depicted, but only symbolized. For similar reasons, at least some diagrams used in analysis may not be depictions either, since they are used to represent infinite processes. I am thankful to Guillermo Zambrana and Jean Dhombres for their helpful insights into the history of diagrams in analysis. I am not fully convinced, however, that arithmetical diagrams used in so-called visual proofs are not depictions. However, the issue deserves more attention than the one I can assign it here.

Renfrew 2009, White 1992, Hodgson 2000, et. al.), where no distinction is made between depictions and other forms of non-linguistic visual representations. Also, notice that even strong advocates of a resemblance account of depiction like Abell (2009), recognize that knowledge of pictorial conventions is necessary for the interpretation of many cases of depiction. Whoever wants to maintain a sharp distinction between symbols and depictions, must explain how the existence of such conventions does not blur the picture/symbol distinction – specially since some pictorial conventions can be as complex and systematic as those governing the semantics of some artificial languages (Greenberg *forthcoming*).

Finally, one must also determine what epistemological consequences about geometrical knowledge and proof can we get from the thesis that geometrical diagrams depict mathematical entities and states of affairs. Is it true, as McCarty claims (unpublished), that the depictive character of diagrams helps us escape the epistemological limitations of mathematical formalism? Are mathematical depictions good news for the realist, and maybe offer a way out of Benacerraff's dilemma, as Brown (2008) suggests? Or is Kitcher (1984) right in claiming that using diagrams in proof threatens its a-priori character? Notice that depiction is not a factive relation. In consequence, we cannot derive the existence of geometrical objects from the fact that geometrical diagrams are depictions. Pictures of non-existent objects are not uncommon, so it might still be the case that the mathematical entities depicted in geometrical diagrams do not exist. In other words, the thesis defended here is amenable to both platonists and nominalists.[30] In general, more work is necessary to fit the thesis that geometrical diagrams are depictions in a broader philosophical account of mathematics.

30. This is not a problem for resemblance based accounts of depiction, which can make sense of this sort of depiction in several ways (Hopkins 1994). For example, it can amend its notion of visual resemblance so that a picture can be said to resemble a non-existent object if it shares some of the visual properties it would have if it existed (Abell 2009).

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